Computer aided verification

Lecture 6:

CTL symbolic model-checking

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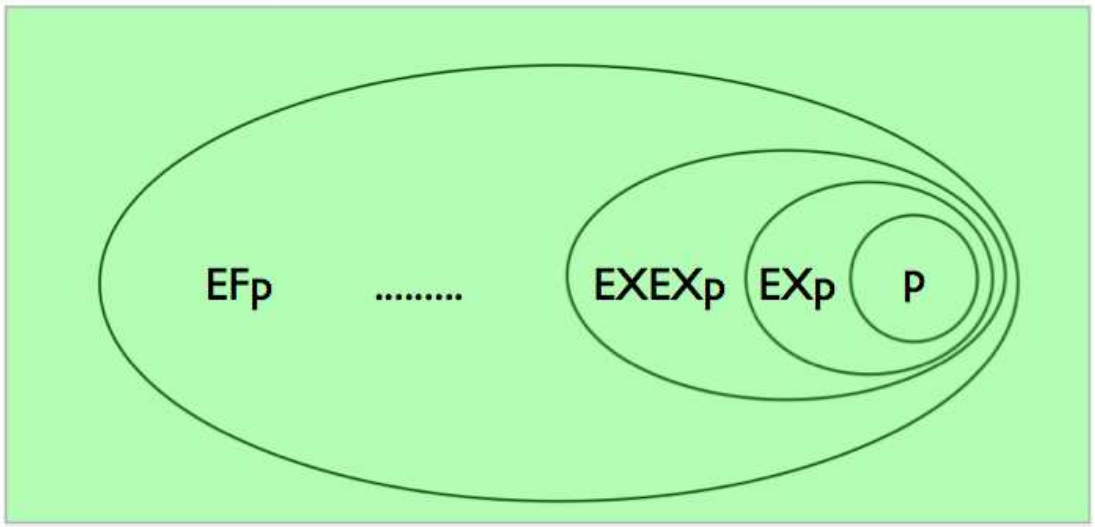
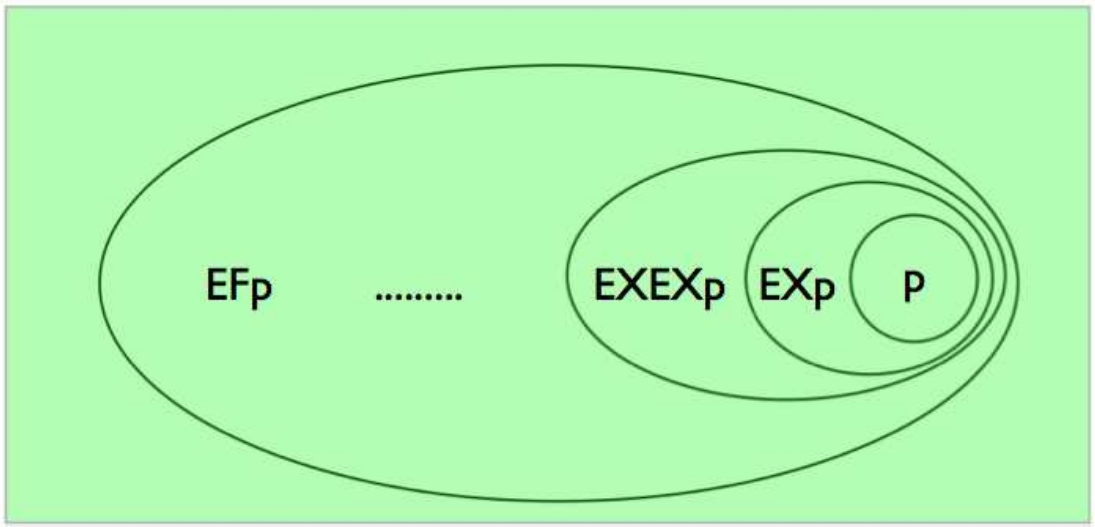
Idea of SMC: EF p



– Compute the set of all states satisfying EF p by iterating predecessor EX _.

Idea of SMC: EF p

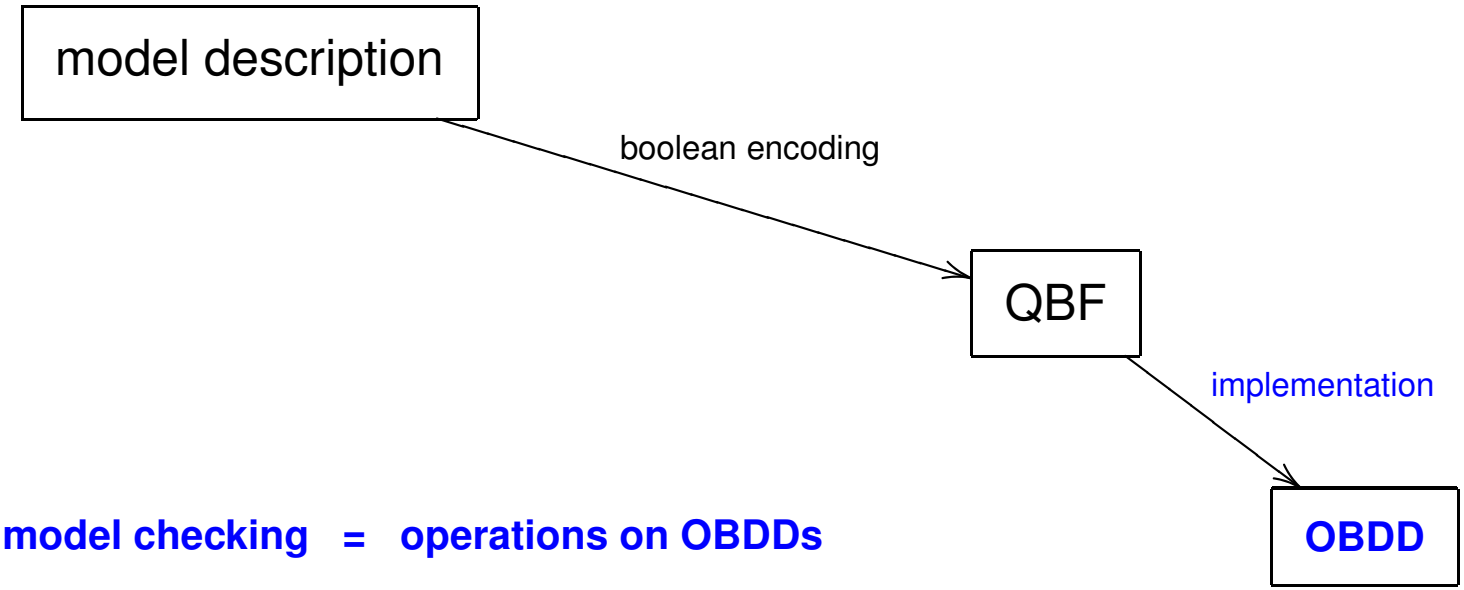


– Compute the set of all states satisfying EF p by iterating predecessor EX _.

– Instead of enumerating all states individually...

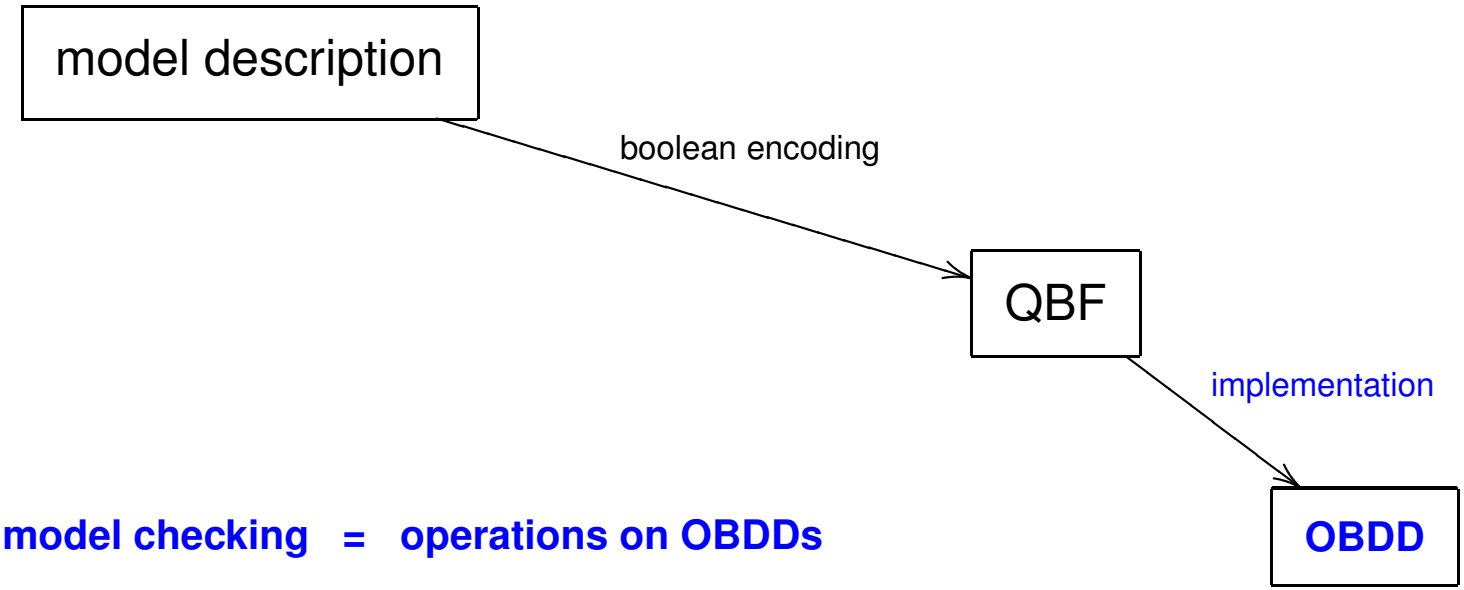
compute **symbolic description** of the set

Symbolic model checking



model checking = operations on OBDDs

Symbolic model checking

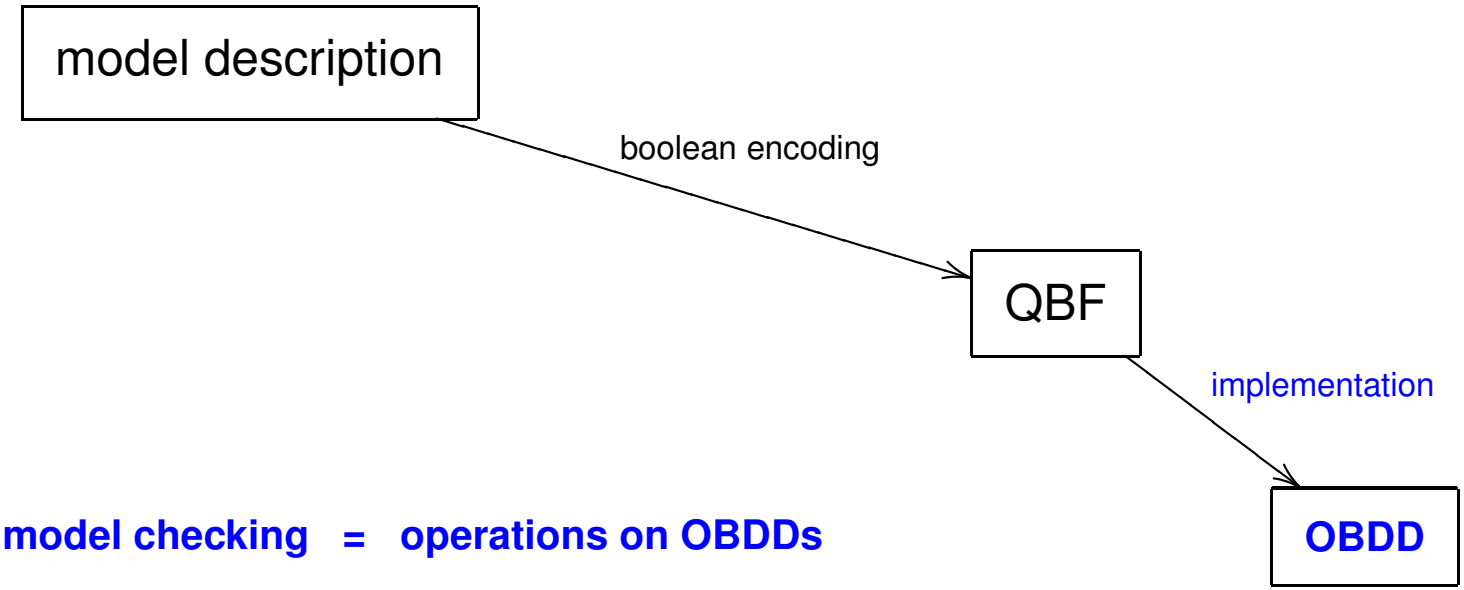


model checking = operations on OBDDs

Plan:

- OBDDs

Symbolic model checking

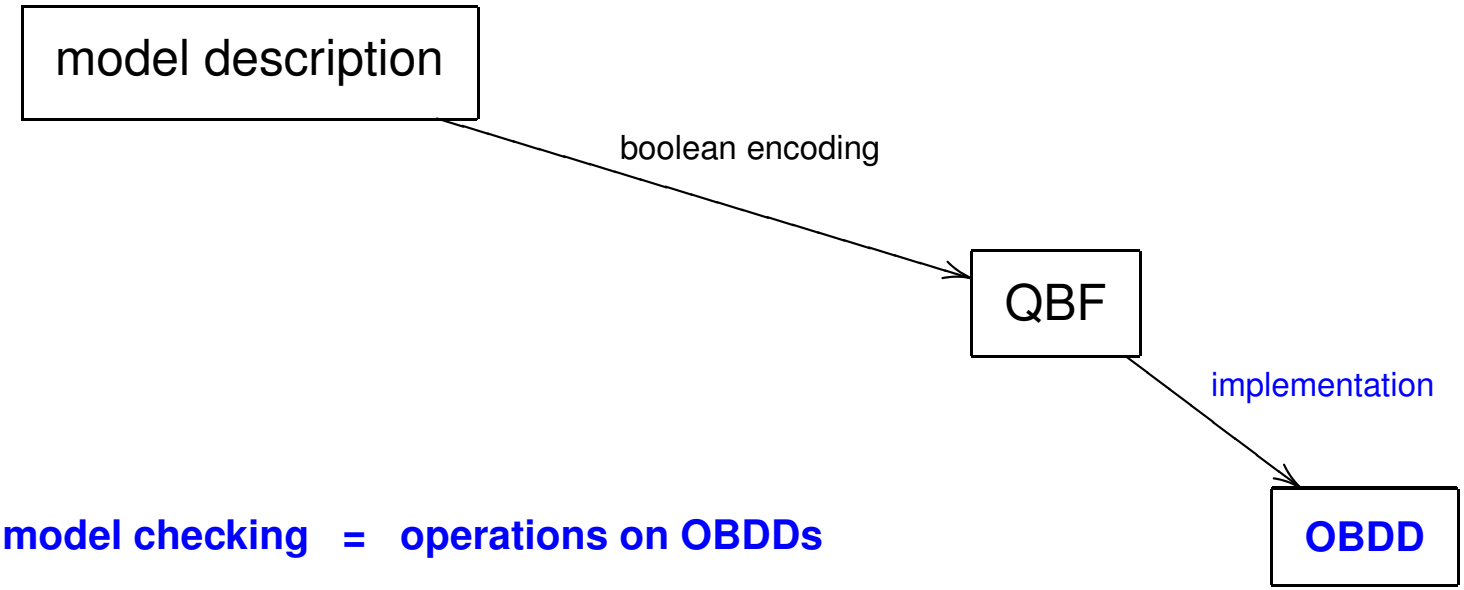


model checking = operations on OBDDs

Plan:

- OBDDs
- Boolean encoding

Symbolic model checking



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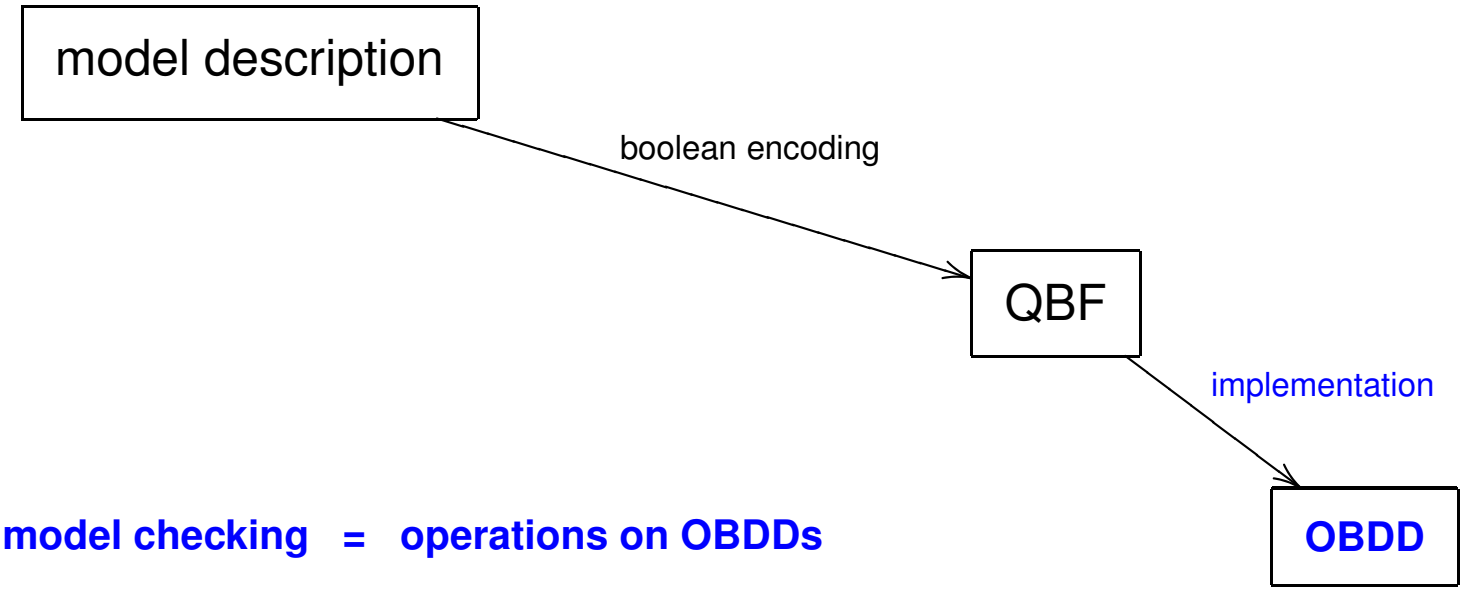
Symbolic model checking



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Plan:

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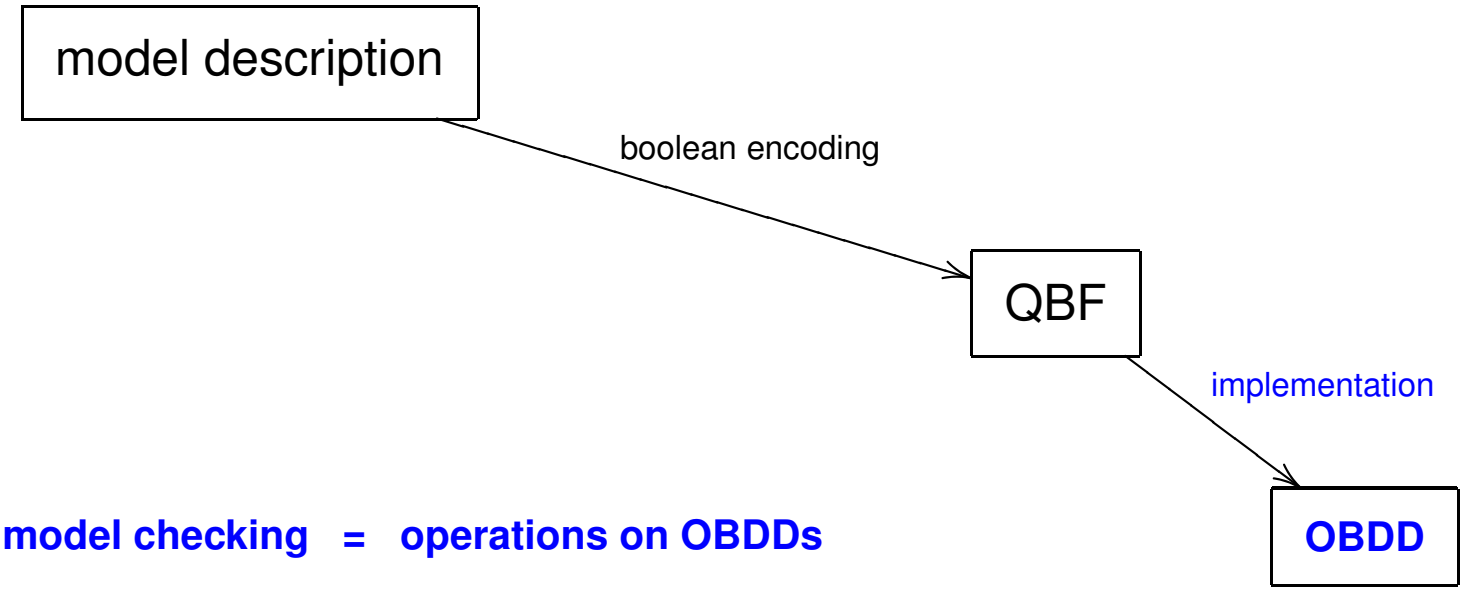
- Boolean encoding
- CTL Symbolic model-checking
- Fairness

Symbolic model checking



model checking = operations on OBDDs

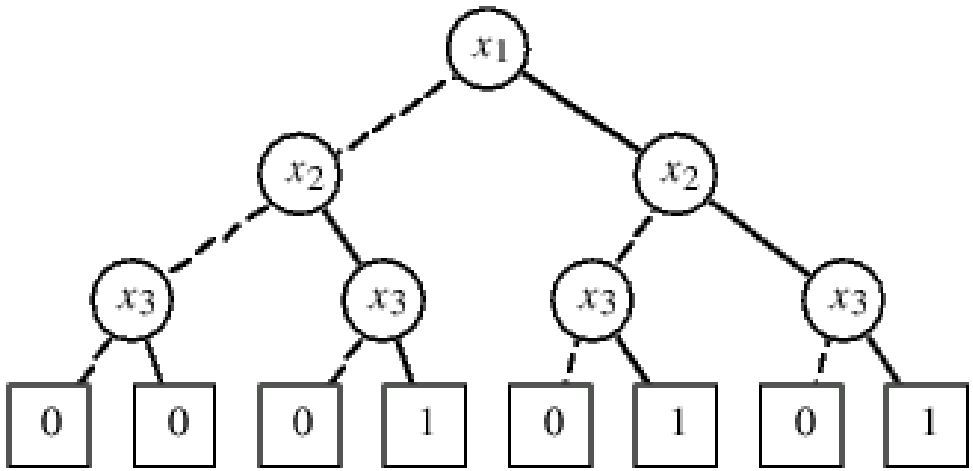
Plan:

- OBDDs
- Boolean encoding
- CTL Symbolic model-checking
- Fairness
- How to compute EX f ?

OBDDs

Ordered Binary Decision Trees

x_1	x_2	x_3	f
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1



[Bryant 1992]

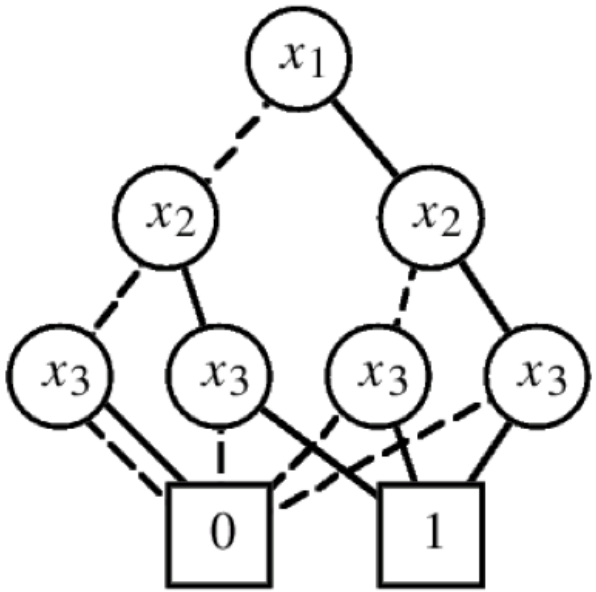
- $f : \{0,1\}^3 \rightarrow \{0,1\}$
- $Z \subseteq \{0,1\}^3$
- fixed order on variables: $x_1 < x_2 < x_3$

Ordered Binary Decision Diagrams

OBDD = rooted directed acyclic graph

Attributes of a node v:

- when v is a leaf
  - a value $\mathrm{val}(v) \in \{0, 1\}$
- when v is not a leaf
  - a variable $\mathrm{var}(v) \in \{x_1, x_2, \ldots\}$
  - 2 successor nodes $s_0(v), s_1(v)$



[Bryant 1992]

The order of variables must be obeyed on every path.

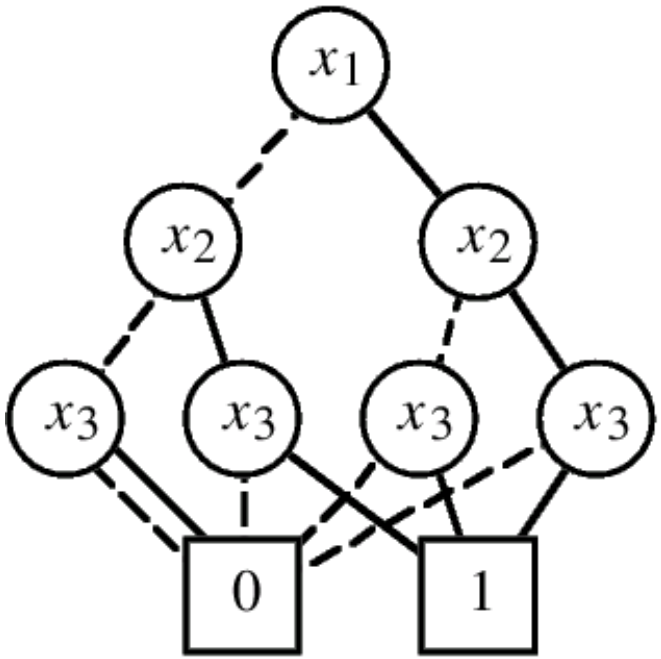
Reduced OBDD

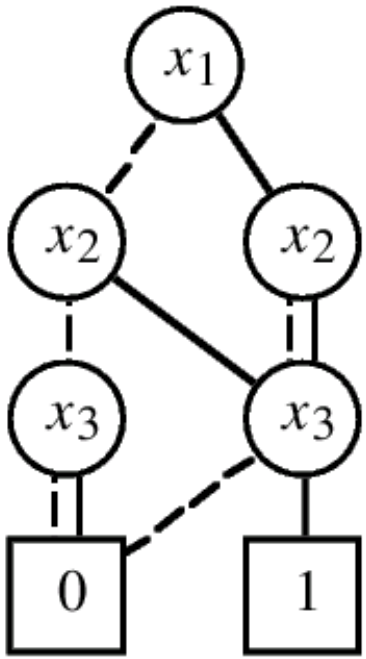
– remove redundant leaves

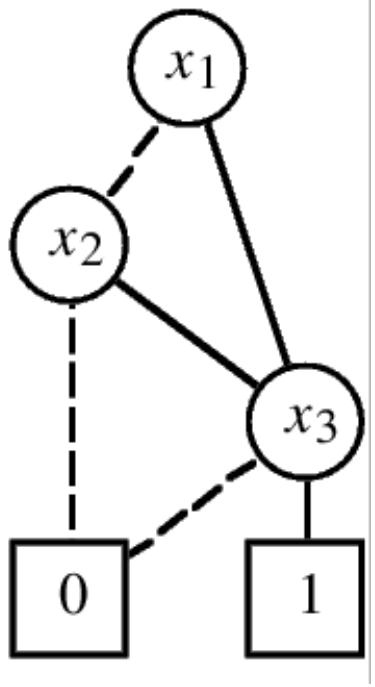
– remove redundant non-leaves

– remove redundant tests

OBDD $\longmapsto$ ROBDD







[Bryant 1992]

Reduced OBDD = canonical form

Canonical form for a boolean function:

- independent from the initial OBDD
- (strongly) dependent on the order of variables

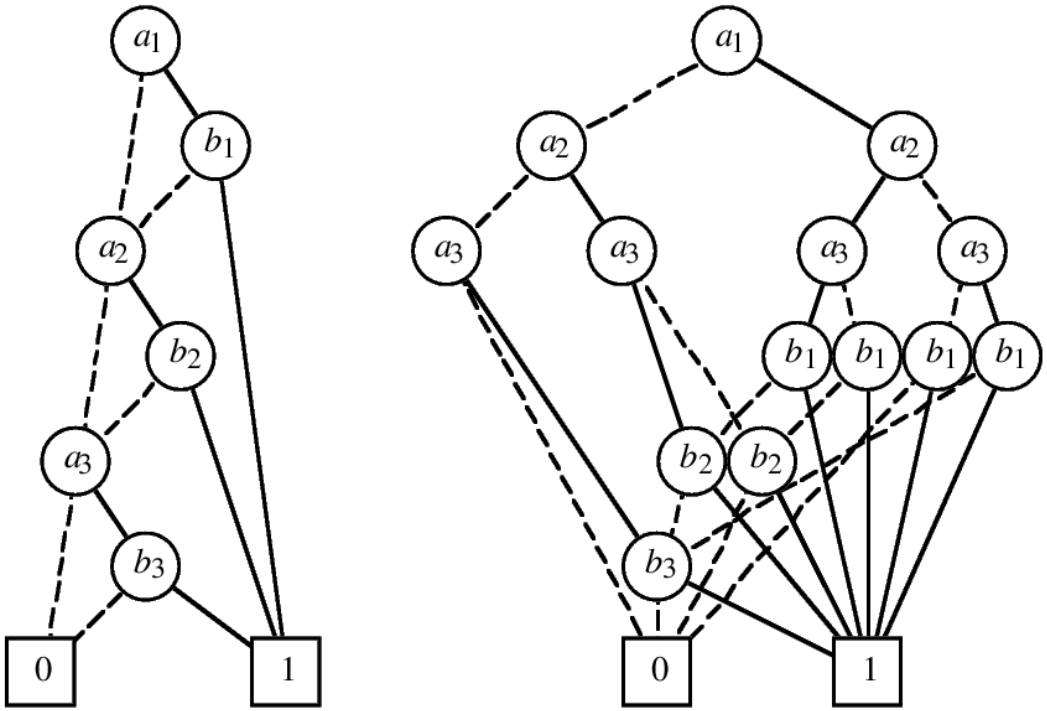
Naive construction of an OBDD for a boolean formula ϕ:

$$\phi \longmapsto \text{decision tree} \longmapsto \text{canonical OBDD}$$

Finding an appropriate order of variables is crucial!

Order of variables is crucial

$$a_1 \wedge b_1 \ \vee \ a_2 \wedge b_2 \ \vee \ a_3 \wedge b_3$$



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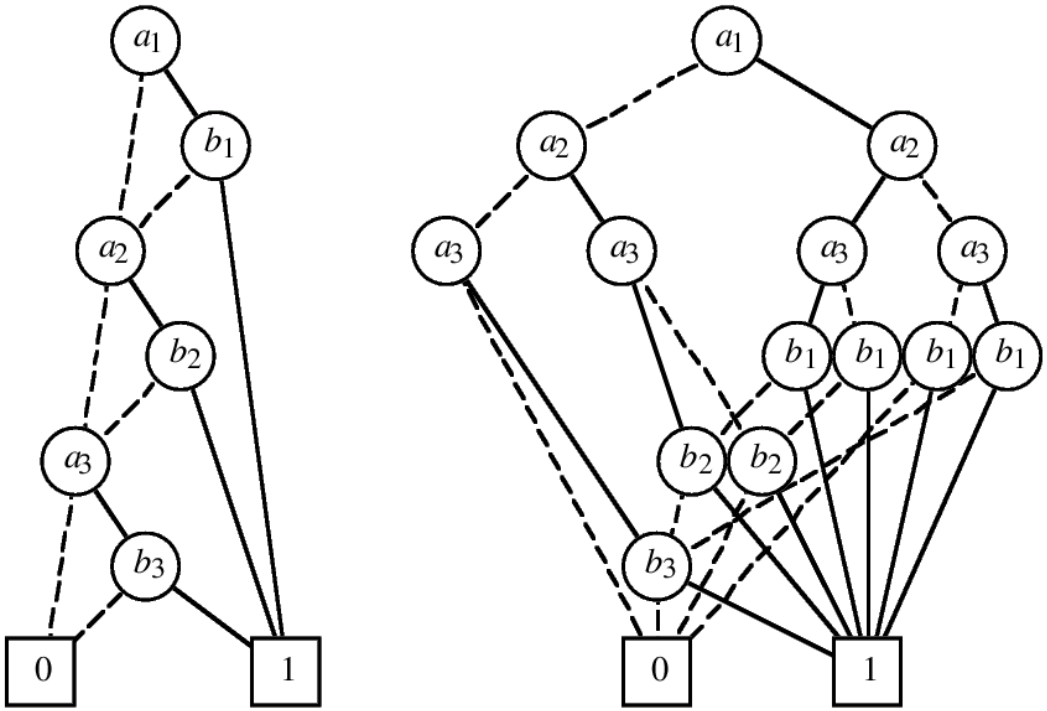

$$a_1 \wedge b_1 \ \vee \ a_2 \wedge b_2 \ \vee \ \ldots \ \vee \ a_n \wedge b_n$$

$2 \cdot n$ $\qquad\qquad$ $2 \cdot (2^n - 1)$

OBDD size

Heuristic: closely related variables should be close in the order

example of a boolean function	lower bound	upper bound
symmetric functions	$\mathcal{O}(n)$	$\mathcal{O}(n^2)$
addition (oldest bits)	$\mathcal{O}(n)$	$\mathcal{O}(2^n)$
multiplication (middle bits)	$\mathcal{O}(2^n)$	$\mathcal{O}(2^n)$

Shannon's expansion

$$f \;=\; \neg x \wedge f\big|_{x \leftarrow 0} \;\vee\; x \wedge f\big|_{x \leftarrow 1}$$

$$f\big|_{x_i \leftarrow b}(x_1, \ldots, x_n) \;=\; f(x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_n)$$

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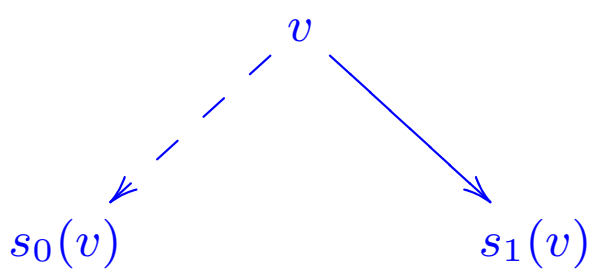
$$v \;=\; \neg x \wedge s_0(v) \;\vee\; x \wedge s_1(v)$$



OBDDs as an abstract data type

the order of variables the same in all OBDDs

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Operations:

$f \vee g$, $f \wedge g$, $\neg f$, `false`, `true` $\qquad BF \mapsto OBDD$

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Note: operations on **boolean functions**, not on values $\{0, 1\}$.

Implementation of unary operations

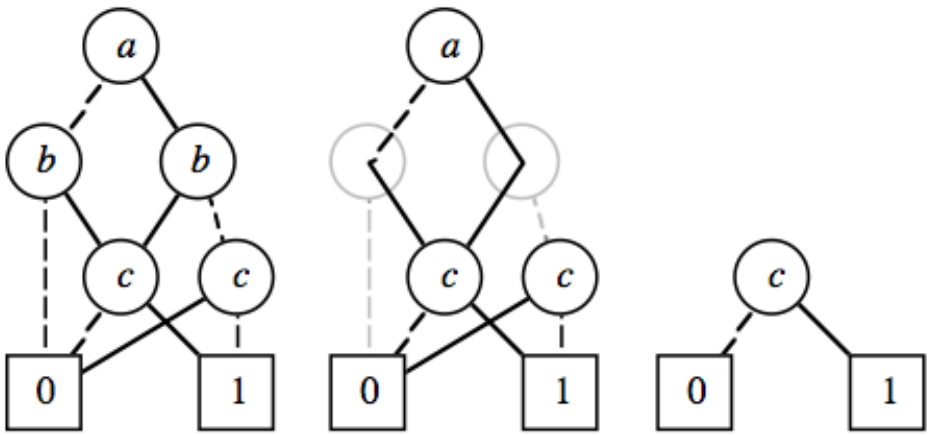
– $f|_{x \leftarrow b}$

traverse nodes n of OBDD representing f:

– if $x > \mathrm{var}(n)$ then recursively traverse $s_0(n)$ and $s_1(n)$;

– if $x < \mathrm{var}(n)$ then stop;

– otherwise replace n by: $\begin{cases} s_0(n) & \text{if } b = 0 \\ s_1(n) & \text{if } b = 1 \end{cases}$



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Implementation of unary operations (cont.)

– $\exists x.\ f \quad = \quad f|_{x \leftarrow 0} \vee f|_{x \leftarrow 1}$

the order of variables

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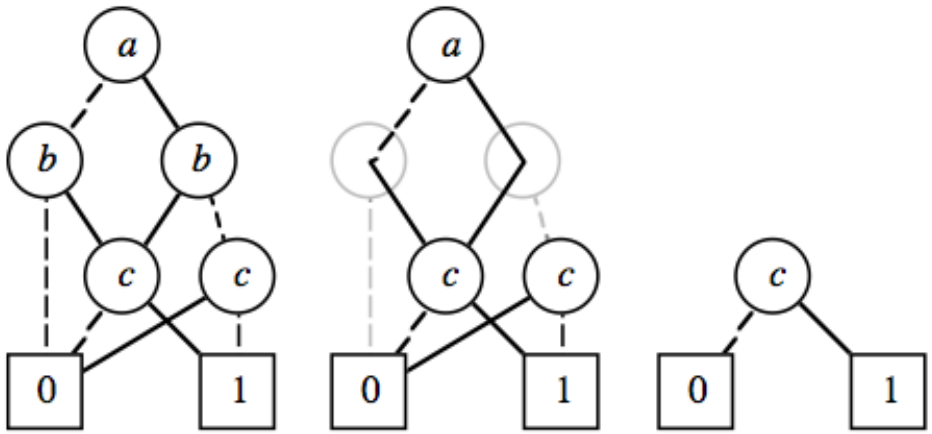
– $\neg f$ **?**



[Bryant 1992]

Implementation of binary operations

the order of variables the same in all OBDDs

$$\bullet : \{0,1\}^2 \to \{0,1\}$$

for two given OBDDs representing f and g,
compute an OBDD representing $f \bullet g$

$$f \bullet g = \neg x \wedge (f \bullet g)|_{x \leftarrow 0} \quad \vee \quad x \wedge (f \bullet g)|_{x \leftarrow 1}$$

$$f \bullet g = \neg x \wedge (f|_{x \leftarrow 0} \bullet g|_{x \leftarrow 0}) \quad \vee \quad x \wedge (f|_{x \leftarrow 1} \bullet g|_{x \leftarrow 1})$$

Implementation of binary operations

$\mathrm{Apply}(\ f,\ g,\ \bullet\ ) \rightsquigarrow\ f \bullet g$ (think of f, g as roots of OBDDs)

- f, g leaves: $\mathrm{val}(f \bullet g) := \mathrm{val}(f) \bullet \mathrm{val}(g)$
- f leaf, g not: $f \bullet g := \mathit{op}(g)$
- $\mathrm{var}(f) = \mathrm{var}(g) = x$:

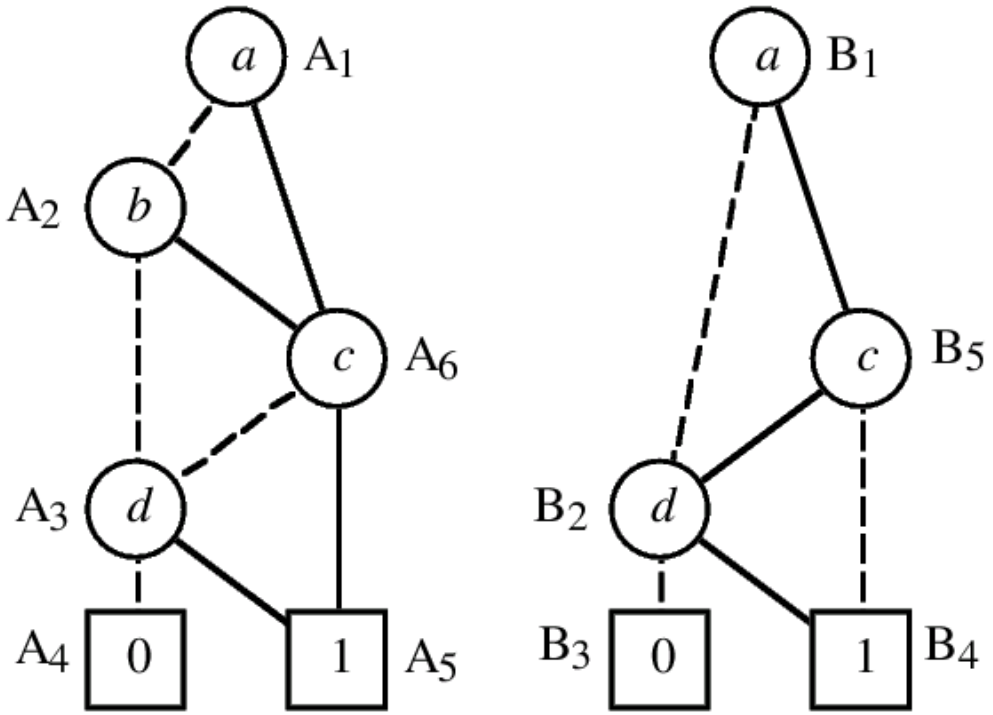
  $\mathrm{var}(f \bullet g) := x \quad s_0(f \bullet g) := s_0(f) \bullet s_0(g) \quad s_1(f \bullet g) := s_1(f) \bullet s_1(g)$
- $\mathrm{var}(f) = x < y = \mathrm{var}(g)$:

  $\mathrm{var}(f \bullet g) := x \qquad s_0(f \bullet g) := s_0(f) \bullet g \qquad s_1(f \bullet g) := s_1(f) \bullet g$

$$f \bullet g = \neg x \wedge (f \bullet g)|_{x \leftarrow 0} \quad \vee \quad x \wedge (f \bullet g)|_{x \leftarrow 1}$$

$$f \bullet g = \neg x \wedge (f|_{x \leftarrow 0} \bullet g|_{x \leftarrow 0}) \quad \vee \quad x \wedge (f|_{x \leftarrow 1} \bullet g|_{x \leftarrow 1})$$

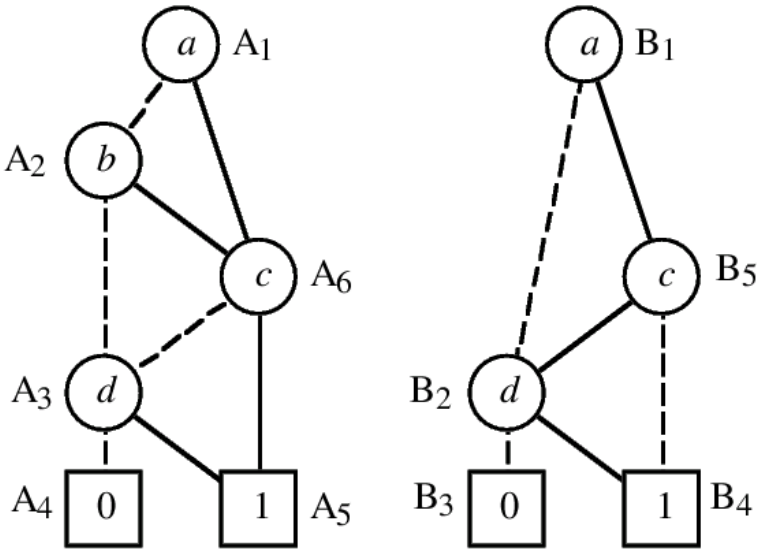
Example: input OBDDs

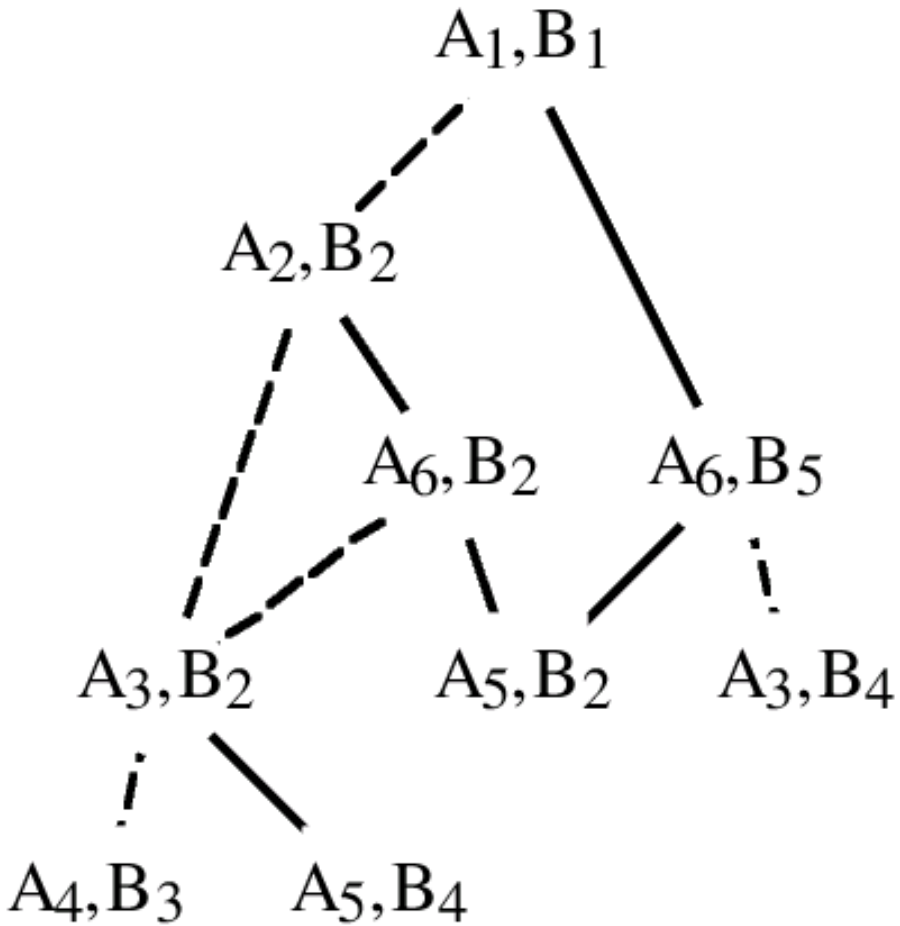


[Bryant 1992]

$(a \vee b) \wedge c \vee d$ $\quad$ $a \wedge \neg c \vee d$

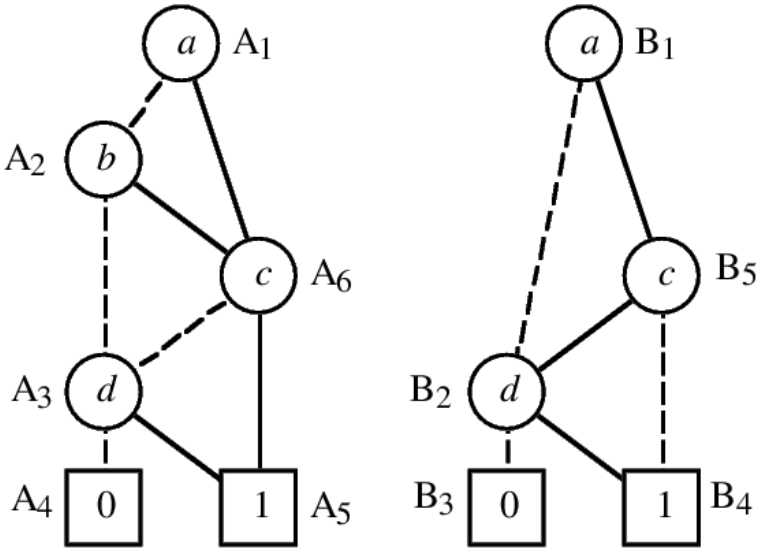
Example: recursive calls

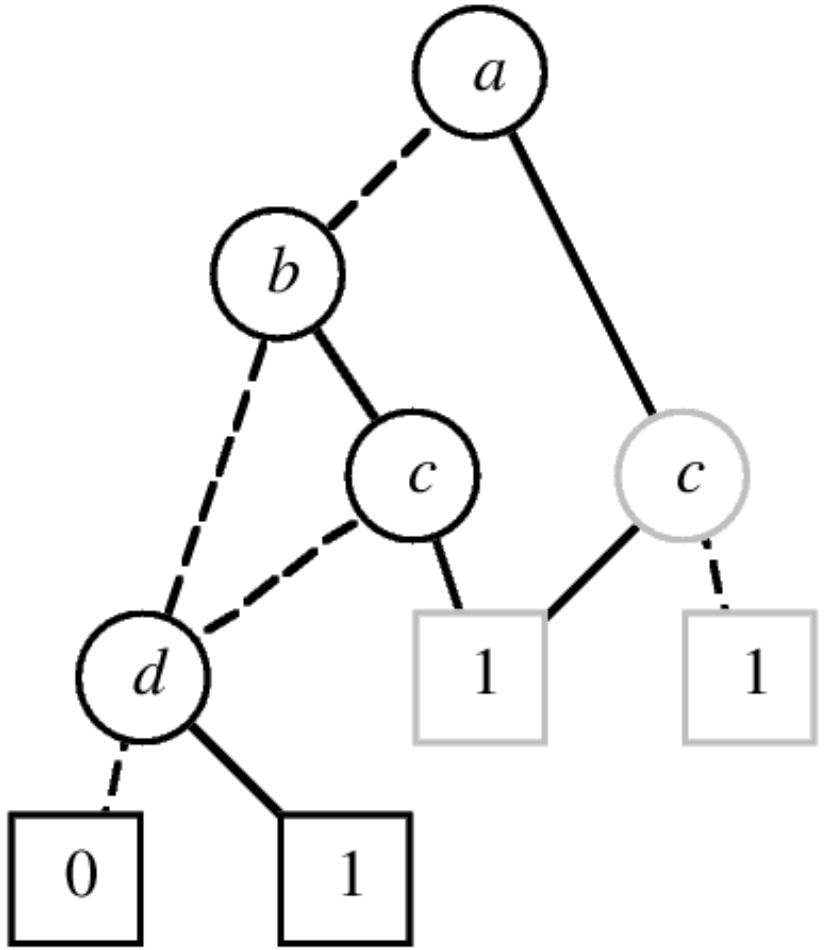




[Bryant 1992]

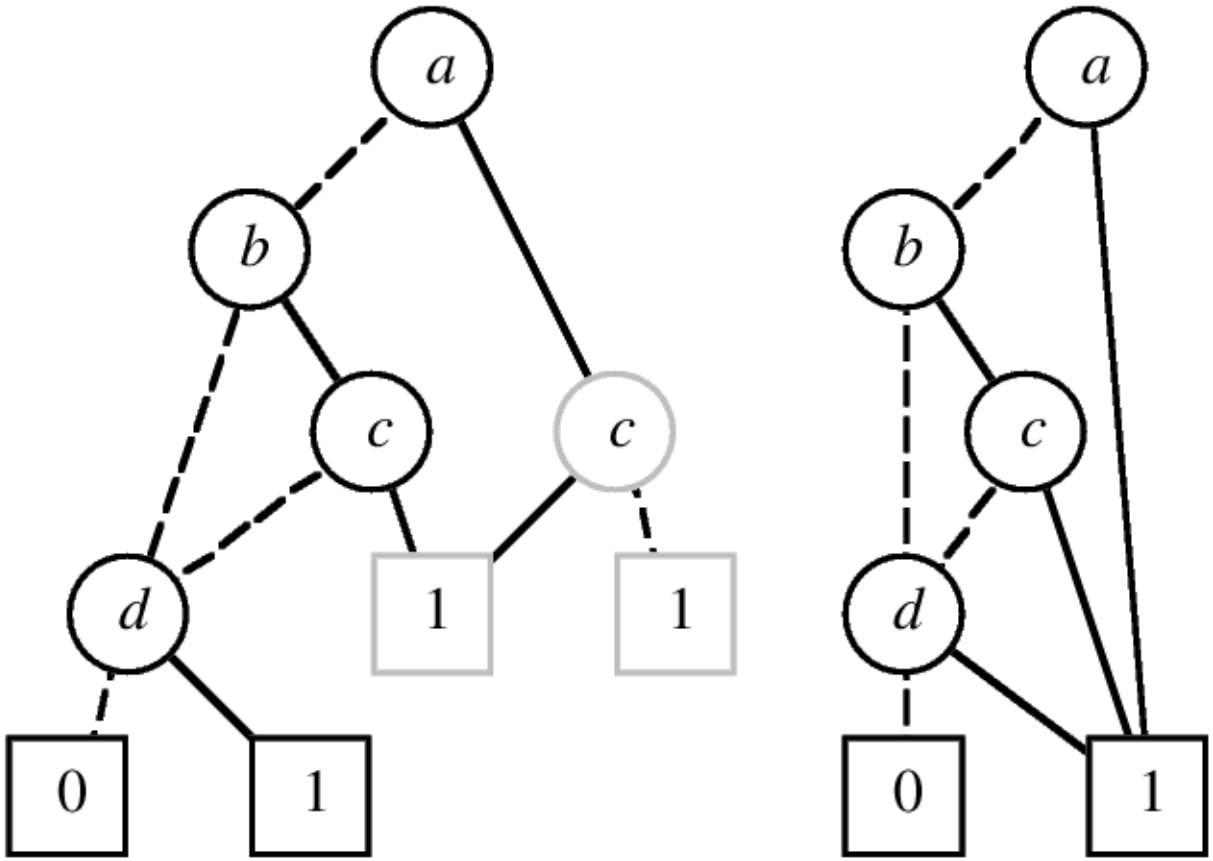
Example: result = $a \vee b \wedge c \vee d$





[Bryant 1992]

Example: reduced result = $a \vee b \wedge c \vee d$



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Implementation of binary operations

$\mathrm{Apply}(\ f,\ g,\ \bullet\ )$

- running time: $\mathcal{O}(|f| \cdot |g|)$
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Question: $f = g$ **?**

Variations and extensions

– one OBDD shared by all functions $\mapsto$ $f = g$ in constant time

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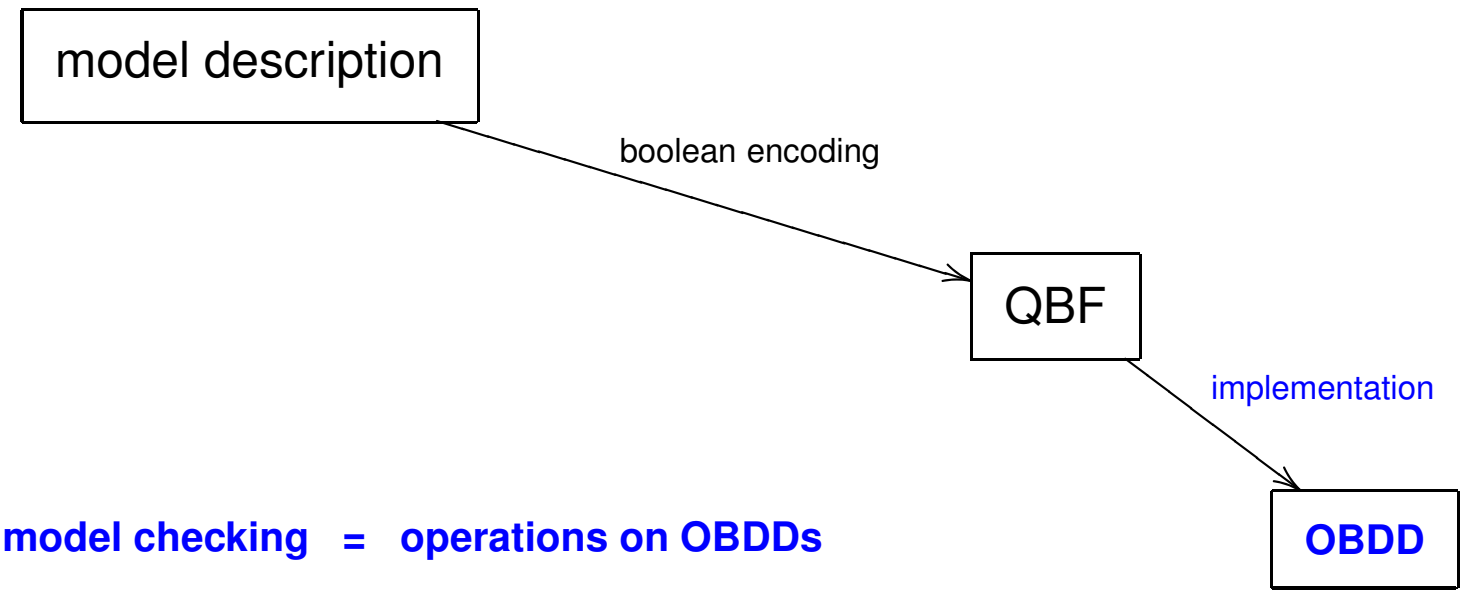
Boolean encoding



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Model description

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– S described by m boolean variables: $S \equiv \{0,1\}^m$

– transition relation $R : \{0,1\}^m \times \{0,1\}^m \to \{0,1\}$

$R(x_1, \ldots, x_m, x'_1, \ldots, x'_m) \in \{0,1\}$

Model description

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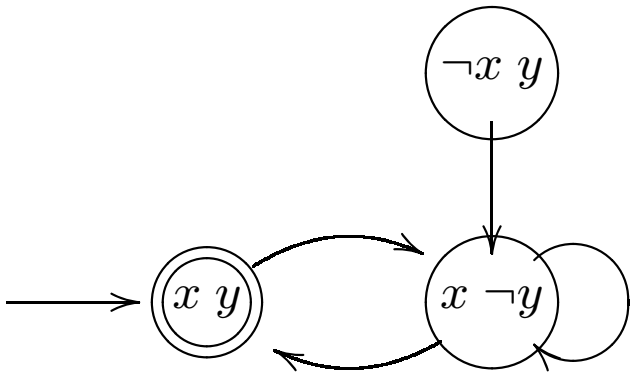
– initial states $S_0 : \{0,1\}^m \to \{0,1\}$

$S_0(x_1, \ldots, x_m) \in \{0,1\}$

– atomic properties $L_p = \{s \mid p \in L(s)\} : \{0,1\}^m \to \{0,1\}$

$L_p(x_1, \ldots, x_m) \in \{0,1\}$

Example



(reachable states)

$$
\begin{aligned}
\text{variables} &= \{x, y\} \\
R &= (x \wedge y \wedge x' \wedge \neg y') \vee (x \wedge \neg y \wedge x' \wedge \neg y') \vee (x \wedge \neg y \wedge x' \wedge y') \\
S_0 &= x \wedge y \\
L_p &= y
\end{aligned}
$$

SMV model – example

```
MODULE main
VAR
  request : {Tr, Fa};
  state : {ready, busy};
ASSIGN
  init(state) := ready;
  next(state) := case
                   state = ready & (request = Tr): busy;
                   TRUE : {ready,busy};
                 esac;
SPEC
  AG((request = Tr) -> AF state = busy)
```

From a model to OBDD

description of a Kripke structure

NO!

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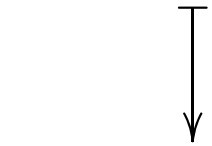

Kripke structure $\longmapsto$ OBDD

NO!

description of a Kripke structure $\longmapsto$ OBDD

YES!

Compositional model desciption

Synchronous processes:

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$$R \;=\; R'_1 \;\vee\; R'_2 \vee \;\ldots\; \vee\; R'_n$$

$$R'_i \;=\; R_i \wedge (\textstyle\bigwedge_{j \neq i} \mathrm{Id}_j)$$

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$$R'_i \;=\; R_i \wedge (\textstyle\bigwedge_{j \neq i} \mathrm{Id}_j)$$

Asynchronous processes (simultaneous execution):

$$R \;=\; R'_1 \;\wedge\; R'_2 \wedge \;\ldots\; \wedge\; R'_n$$

$$R'_i \;=\; R_i \vee \mathrm{Id}_i$$

Restriction to reachable states

$$\widehat{R}(x_1, \ldots, x_m, x'_1, \ldots, x'_m) = 1$$

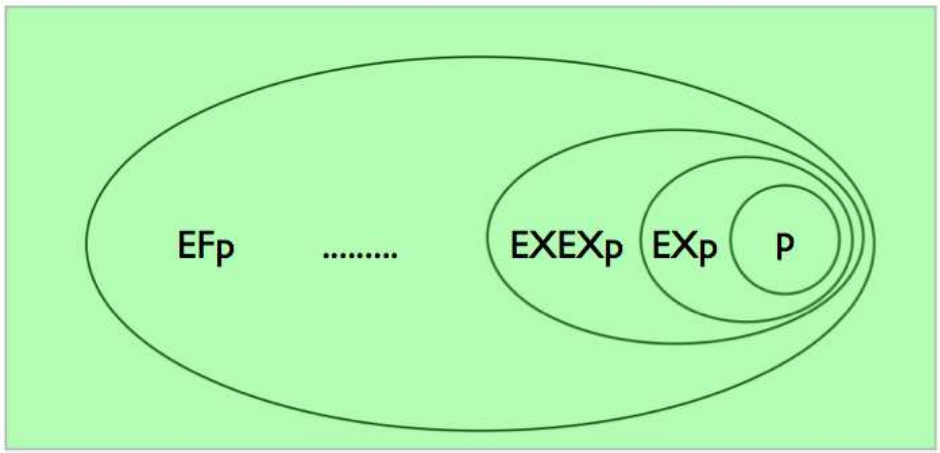
$$\Updownarrow$$

$$R(x_1, \ldots, x_m, x'_1, \ldots, x'_m) \ \wedge$$

$$(x_1, \ldots, x_m) \text{ reachable}$$

Should we restrict to reachable states?

CTL Symbolic model-checking



Reminder

Fixed points in a complete lattice $\langle A, \leq \rangle$. Assume A is finite.

Let $f : A \to A$ monotonic.

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Let $f : A \to A$ monotonic.

- the least f.p.: $\bot \leq f(\bot) \leq f^2(\bot) \leq \ldots \rightsquigarrow \mu Z.\, f(Z)$
- the greatest f.p.: $\top \geq f(\top) \geq f^2(\top) \geq \ldots \rightsquigarrow \nu Z.\, f(Z)$

When A finite, the fixed points are reached after $\leq |A|$ iterations.

Reminder

Fixed points in a complete lattice $\langle A, \leq \rangle$. Assume A is finite.

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$Z \mapsto$ EX Z $\quad$ $\mu Z.$ EX $Z = \bot = \emptyset$ $\quad$ $\nu Z.$ EX Z = **?**

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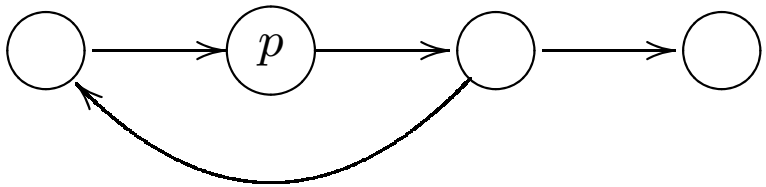
$Z \mapsto$ EX Z $\quad$ $\mu Z.$ EX $Z = \bot = \emptyset$ $\quad$ $\nu Z.$ EX Z = ?

$Z \mapsto p \vee$ EX Z $\quad$ $\mu Z.\, p \vee$ EX Z = ?

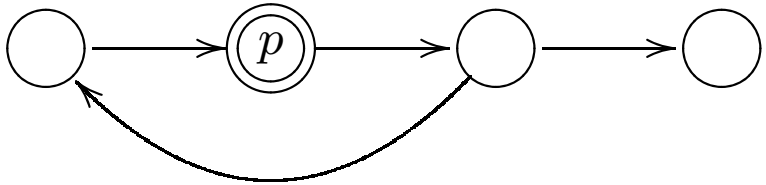
Example

$\mathsf{EF}\, p \;=\; \mu Z.\, p \vee \mathsf{EX}\, Z$ $\qquad\qquad$ $Z \mapsto p \vee \mathsf{EX}\, Z$

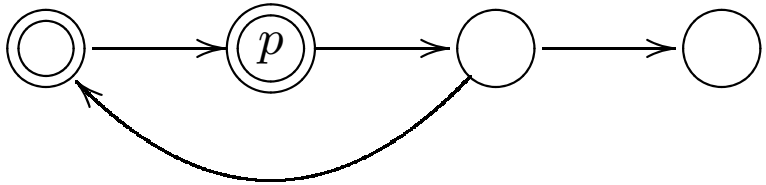
$\mathtt{false}$



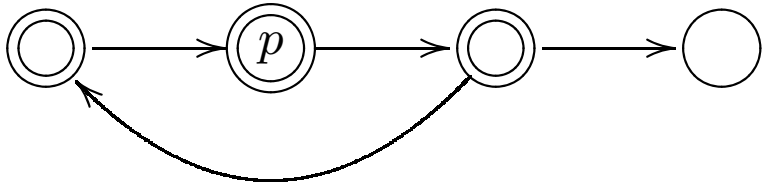
$p \vee \mathsf{EX}\, \mathtt{false} \equiv p$



$p \vee \mathsf{EX}\, p$



$p \vee \mathsf{EX}\, (p \vee \mathsf{EX}\, p)$



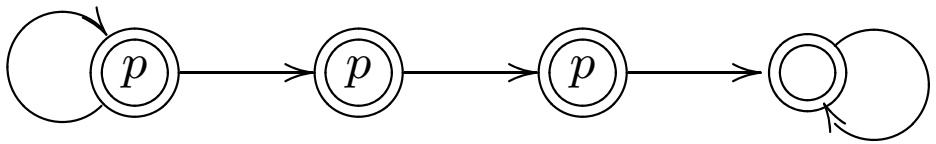
CTL via fixed points

- $\mathsf{EF}\,\phi \;=\; \mu Z.\, \phi \vee \mathsf{EX}\, Z$ $\qquad Z \mapsto \phi \vee \mathsf{EX}\, Z$
- $\mathsf{AF}\,\phi \;=\; \mu Z.\, \phi \vee \mathsf{AX}\, Z$ $\qquad Z \mapsto \phi \vee \mathsf{AX}\, Z$
- $\mathsf{EG}\,\phi \;=\; \nu Z.\, \phi \wedge \mathsf{EX}\, Z$ $\qquad Z \mapsto \phi \wedge \mathsf{EX}\, Z$
- $\mathsf{AG}\,\phi \;=\; \nu Z.\, \phi \wedge \mathsf{AX}\, Z$ $\qquad Z \mapsto \phi \wedge \mathsf{AX}\, Z$
- $\mathsf{E}\,\phi\,\mathsf{U}\,\psi \;=\; \mu Z.\, \psi \vee (\phi \wedge \mathsf{EX}\, Z)$ $\qquad Z \mapsto \psi \vee (\phi \wedge \mathsf{EX}\, Z)$
- $\mathsf{A}\,\phi\,\mathsf{U}\,\psi \;=\; \mu Z.\, \psi \vee (\phi \wedge \mathsf{AX}\, Z)$ $\qquad Z \mapsto \psi \vee (\phi \wedge \mathsf{AX}\, Z)$
- ...

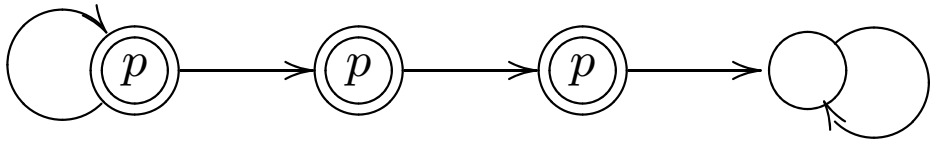
Example

$\mathsf{EG}\, p \;=\; \nu Z.\, p \wedge \mathsf{EX}\, Z \qquad\qquad Z \mapsto p \wedge \mathsf{EX}\, Z$

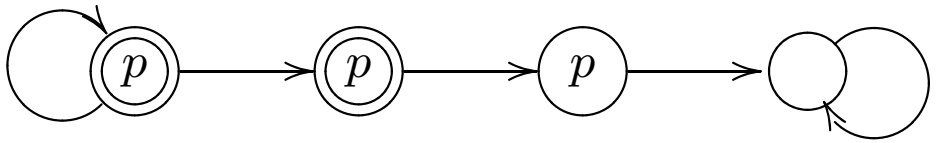
$\texttt{true}$



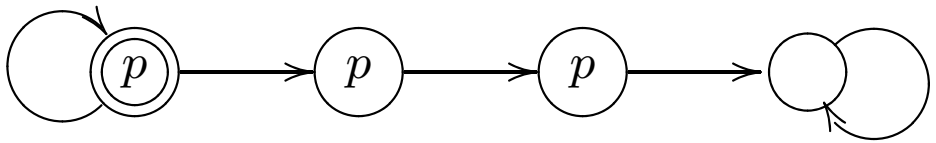
$p \wedge \mathsf{EX}\, \texttt{true} \equiv p$



$p \wedge \mathsf{EX}\, p$



$p \wedge \mathsf{EX}\, (p \wedge \mathsf{EX}\, p)$



CTL Symbolic model checking

CTL $(\neg, \wedge,$ **EX**, **E_U_**, **EG**$)$ (these connectives are sufficient)

$\mathrm{OBDD} : \mathrm{CTL} \mapsto \mathrm{OBDD}$

$\mathrm{OBDD}(\phi)$ represents $\{s \mid s \vDash \phi\}$

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– $\mathrm{OBDD}(p)$:= OBDD representing L_p

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- $\mathrm{OBDD}(p) :=$ OBDD representing L_p
- $\mathrm{OBDD}(\neg\phi) := \neg\mathrm{OBDD}(\phi)$

CTL Symbolic model checking

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- $\mathrm{OBDD}(p) := $ OBDD representing L_p
- $\mathrm{OBDD}(\neg\phi) := \neg\mathrm{OBDD}(\phi)$
- $\mathrm{OBDD}(\phi \wedge \psi) := \mathrm{OBDD}(\phi) \wedge \mathrm{OBDD}(\psi)$

Symbolic model checking (EX _)

$\mathrm{OBDD} : \mathrm{CTL} \to \mathrm{OBDD}$

$\mathrm{OBDD}(\phi)$ represents $\{s \mid s \vDash \phi\}$

$\mathrm{OBDD}(\mathsf{EX}\,\phi) \;:=\; \exists \vec{x}'.\, R(\vec{x}, \vec{x}') \wedge f[\vec{x}'/\vec{x}]$ where $f = \mathrm{OBDD}(\phi)$

$\mathrm{OBDD}(\mathsf{EX}\,\phi) \;:=\; \mathsf{EX}\, f \;=\; \mathsf{EX}\,\mathrm{OBDD}(\phi)$

$\mathsf{EX}\,\phi$

$\mathsf{EX}\, Z$

$\mathsf{EX}\, f$

Order of variables

$$\exists \vec{x}'.\ R(\vec{x}, \vec{x}') \wedge f[\vec{x}'/\vec{x}])$$

$$\vec{x} = x_1, x_2, \ldots, x_m$$

$$x_1 < x_2 < \ldots < x_m$$

$$x_1 < x_1' < x_2 < x_2' < \ldots < x_m < x_m'$$

$x_i < x_j$ if and only if $x_i' < x_j'$

Symbolic model checking (E _ U _)

$\mathrm{OBDD} : \mathrm{CTL} \to \mathrm{OBDD}$

$\mathrm{OBDD}(\phi)$ represents $\{s \mid s \vDash \phi\}$

$\mathrm{OBDD}(\mathsf{E}\,\phi\,\mathsf{U}\,\psi) \;:=\; \mu Z.\; g \vee (f \wedge \mathsf{EX}\, Z)$

where $f = \mathrm{OBDD}(\phi)$
$g = \mathrm{OBDD}(\psi)$

$h \mapsto g \vee (f \wedge \mathsf{EX}\, h)$

$h \mapsto g \vee (f \wedge \exists \vec{x}'.\; R(\vec{x}, \vec{x}') \wedge h[\vec{x}'/\vec{x}])$

Symbolic model checking (E _ U _)

OBDD : CTL $\to$ OBDD

$\mathrm{OBDD}(\phi)$ represents $\{s \mid s \models \phi\}$

$\mathrm{OBDD}(\mathsf{E}\,\phi\,\mathsf{U}\,\psi) \;:=\; \mu Z.\, g \vee (f \wedge \mathsf{EX}\, Z)$

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$$h \mapsto g \vee (f \wedge \mathsf{EX}\, h)$$

$$h \mapsto g \vee (f \wedge \exists \vec{x}'.\, R(\vec{x}, \vec{x}') \wedge h[\vec{x}'/\vec{x}])$$

Implement computation of the approximations of lfp by OBDDs:

$$\begin{array}{lcll} h_0 & \equiv & \texttt{false} & \\ h_1 & \equiv & g \vee (f \wedge \mathsf{EX}\, h_0) & \equiv \quad g \\ h_2 & \equiv & g \vee (f \wedge \mathsf{EX}\, h_1) & \equiv \quad g \vee (f \wedge \mathsf{EX}\, g) \\ h_3 & \equiv & \ldots & \equiv \quad g \vee (f \wedge \mathsf{EX}\, (g \vee (f \wedge \mathsf{EX}\, g))) \end{array}$$

$$\mu Z.\, g \vee (f \wedge \mathsf{EX}\, Z) \qquad (h_i = h_{i+1})$$

Symbolic model checking (EG _)

OBDD : CTL $\to$ OBDD

$\mathrm{OBDD}(\phi)$ represents $\{s \mid s \vDash \phi\}$

$\mathrm{OBDD}(\mathsf{EG}\,\phi) \;:=\; \nu Z.\, f \wedge \mathsf{EX}\, Z$ where $f = \mathrm{OBDD}(\phi)$

$h \mapsto f \wedge \mathsf{EX}\, h$

$h \mapsto f \wedge \exists \vec{x}'.\, R(\vec{x}, \vec{x}') \wedge h[\vec{x}'/\vec{x}]$

Symbolic model checking (EG _)

OBDD : CTL $\to$ OBDD

$\mathrm{OBDD}(\phi)$ represents $\{s \mid s \vDash \phi\}$

$\mathrm{OBDD}(\mathsf{EG}\,\phi) \quad := \quad \nu Z.\, f \wedge \mathsf{EX}\, Z$ where $f = \mathrm{OBDD}(\phi)$

$$h \mapsto f \wedge \mathsf{EX}\, h$$

$$h \mapsto f \wedge \exists \vec{x}'.\, R(\vec{x}, \vec{x}') \wedge h[\vec{x}'/\vec{x}]$$

Implement computation of the approximations of gfp by OBDDs, as bofore.

EX ϕ	E ϕ U ψ	EG ϕ
EX Z	E Z U Z'	EG Z
EX f	E f U g	EG f

Graph diameter

Efficiency of SMC depends on the diameter of M.

Fairness

Fair CTL

$\mathbf{F} = \{\psi_1, \ldots, \psi_n\}, \quad \psi_i \in \mathsf{CTL} \qquad \mapsto \quad F = \{Z_1, \ldots, Z_n\}$

$$
\begin{array}{lcl}
s \models_{\mathbf{F}} p & \iff & p \in L(s) \ \wedge \ \exists \text{ fair } \Pi \text{ from } s \\
s \models_{\mathbf{F}} \mathsf{A}\,\phi\,\mathsf{U}\,\psi & \iff & \forall \text{ fair } \Pi \text{ from } s \,.\ \Pi \models \phi\,\mathsf{U}\,\psi \\
s \models_{\mathbf{F}} \mathsf{E}\,\phi\,\mathsf{U}\,\psi & \iff & \exists \text{ fair } \Pi \text{ from } s \,.\ \Pi \models \phi\,\mathsf{U}\,\psi \\
s \models_{\mathbf{F}} \mathsf{AX}\,\phi & \iff & \forall \text{ fair } \Pi \text{ from } s \,.\ \Pi \models \mathsf{X}\,\phi \\
s \models_{\mathbf{F}} \mathsf{EX}\,\phi & \iff & \exists \text{ fair } \Pi \text{ from } s \,.\ \Pi \models \mathsf{X}\,\phi
\end{array}
$$

$\mathbf{F} = \{h_1, \ldots, h_n\}, \quad h_i \in \mathsf{OBDD}$

Fair EG

$\mathbf{F} = \{\psi_1, \ldots, \psi_n\}, \quad \psi_i \in \text{CTL} \qquad \mapsto \quad F = \{Z_1, \ldots, Z_n\}$

$\text{EG}\,\phi \;=\; \{s \mid s \vDash_{\mathbf{F}} \text{EG}\,\phi\} \;=$ the greatest Z s.t. if $s \in Z$ then

- $s \vDash \phi$
- $\forall i \leq n \,.\; \exists s' \,.\; s \rightarrow \ldots \rightarrow s' \in Z_i \cap Z, \quad s' \neq s,$ all intermediate states satisfy ϕ

Fair EG

$\mathbf{F} = \{\psi_1, \ldots, \psi_n\}, \quad \psi_i \in \mathsf{CTL} \qquad \mapsto \quad F = \{Z_1, \ldots, Z_n\}$

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$\mathsf{EG}\,\phi \;=\; \nu Z.\, \phi \wedge \bigwedge_{i=1}^{n} \mathsf{EX}\,\mathsf{E}\,\phi\,\mathsf{U}\,(\psi_i \wedge Z)$

Fair EG

$\mathbf{F} = \{\psi_1, \ldots, \psi_n\}, \quad \psi_i \in \mathsf{CTL} \qquad \mapsto \quad F = \{Z_1, \ldots, Z_n\}$

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$\mathsf{EG}\,\phi \;=\; \nu Z.\, \phi \wedge \bigwedge_{i=1}^{n} \mathsf{EX}\,\mathsf{E}\,\phi\,\mathsf{U}\,(\psi_i \wedge Z)$

$\mathsf{EG}\,\phi \;=\; \nu Z.\, \phi \wedge \bigwedge_{i=1}^{n} \mathsf{EX}\,\mu Y.(\psi_i \wedge Z) \vee (\phi \wedge \mathsf{EX}\,Y)$ alternation!

Fair EG

Thm:

$$\mathsf{EG}\,\phi \;=\; \nu Z.\;\phi \wedge \bigwedge_{i=1}^{n} \mathsf{EX}\,\mathsf{E}\,\phi\,\mathsf{U}\,(\psi_i \wedge Z)$$

Proof:

$$\mathsf{EG}\,\phi \;=\; \phi \wedge \bigwedge_{i=1}^{n} \mathsf{EX}\,\mathsf{E}\,\phi\,\mathsf{U}\,(\psi_i \wedge \mathsf{EG}\,\phi)$$

$$Z \;=\; \phi \wedge \bigwedge_{i=1}^{n} \mathsf{EX}\,\mathsf{E}\,\phi\,\mathsf{U}\,(\psi_i \wedge Z) \implies Z \subseteq \mathsf{EG}\,\phi$$

Fair symbolic model checking (EG _)

$\mathrm{OBDD} : \mathrm{CTL} \to \mathrm{OBDD}$

$\mathrm{OBDD}(\phi)$ represents $\{s \mid s \models_{\mathbf{F}} \phi\}$

$\mathbf{F} = \{\psi_1, \ldots, \psi_n\}, \psi_i \in \mathrm{CTL} \quad \mapsto \quad F = \{h_1, \ldots, h_n\}, h_i \in \mathrm{OBDD}$

$\mathrm{OBDD}(\mathsf{EG}\,\phi) \;:=\; \nu Z.\, f \;\wedge\; \bigwedge_{i=1}^{n} \mathsf{EX}\,\mathsf{E}\, f\, \mathsf{U}\, (h_i \wedge Z)$

where $f = \mathrm{OBDD}(\phi)$

$$Z \mapsto f \wedge \bigwedge_{i=1}^{n} \mathsf{EX}\,\mathsf{E}\, f\, \mathsf{U}\, (h_i \wedge Z)$$

Fair CTL symbolic model checking

$$\texttt{fair} \;:=\; \mathrm{OBDD}(\,\mathsf{EG}\,\texttt{true})$$

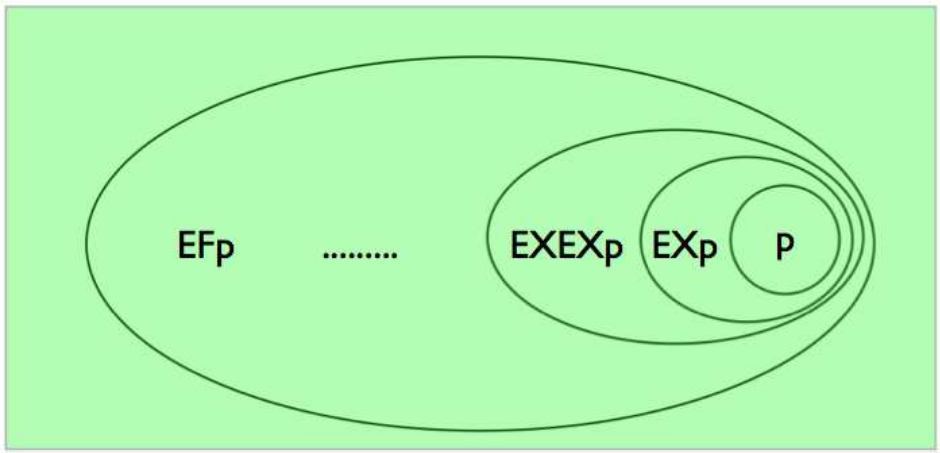
$$\mathrm{OBDD}(\,\mathsf{EX}\,\phi) \;:=\; \exists \vec{x}'.\; R(\vec{x},\vec{x}') \wedge f(\vec{x}') \wedge \texttt{fair}(\vec{x}')$$

where $f = \mathrm{OBDD}(\phi)$

$$\mathrm{OBDD}(\,\mathsf{E}\,\phi\,\mathsf{U}\,\psi) \;:=\; \mu Z.\; (g \wedge \texttt{fair}) \vee (f \wedge \mathsf{EX}\, Z)$$

where $f = \mathrm{OBDD}(\phi)$
$g = \mathrm{OBDD}(\psi)$

How to compute EX f ?



EX f by AndExist

$$\mathsf{EX}\, f \;:=\; \exists \vec{x}'.\, R(\vec{x}, \vec{x}') \wedge f(\vec{x}')$$

operation $\exists\wedge(g, h, V) \;:=\; \exists V.\, g \wedge h$ (V – set of variables)

$R(x_1, \ldots, x_m, x'_1, \ldots, x'_m)$

$f(x_1, \ldots, x_m) \;\mapsto\; f'(x'_1, \ldots, x'_m)$ $\qquad x_i \leq x_j \iff x'_i \leq x'_j$

$$\mathsf{EX}\, f \;=\; \exists\wedge(R, f', \{x'_1, \ldots, x'_m\})$$

AndExist

$\exists\wedge(f,\ g, V)$ $\qquad\qquad$ $(\ \exists V.\ f \wedge g\ )$

- f, g leaves: $\mathrm{val}(\exists\wedge(f,g,V)) := \mathrm{val}(f) \wedge \mathrm{val}(g)$
- f a leaf, g not: $\exists\wedge(f,g,V) :=$ `false` or $\exists V.\ g$
- $x = \mathrm{var}(f) = \mathrm{var}(g)$:

  $l := \exists\wedge(s_0(f), s_0(g), V), \quad h := \exists\wedge(s_1(f), s_1(g), V)$

  - $x \in V$: $\exists\wedge(f,g,V) := l \vee h$
  - $x \notin V$: $s_0(\exists\wedge(f,g,V)) := l$ $\qquad\qquad$ $s_1(\exists\wedge(f,g,V)) := h$
- $x = \mathrm{var}(f) < \mathrm{var}(g)$: ...

$$f \bullet g \ = \ \neg x \wedge (f|_{x\leftarrow 0} \bullet g|_{x\leftarrow 0}) \ \vee \ x \wedge (f|_{x\leftarrow 1} \bullet g|_{x\leftarrow 1})$$

R is not monolytic

$$\mathsf{EX}\, f \;:=\; \exists \vec{x}'.\, R(\vec{x}, \vec{x}') \wedge f(\vec{x}')$$

Asynchronous model: $R \;=\; R'_1 \;\vee\; R'_2 \vee \;\ldots\; \vee\; R'_n$

$$R'_i \;=\; R_i \wedge \textstyle\bigwedge_{j \neq i} \mathrm{Id}_j$$

Synchronous model: $R \;=\; R_1 \;\wedge\; R_2 \wedge \;\ldots\; \wedge\; R_n$

Can one profit from this additional structure ?

Asynchronous model: $R = R'_1 \vee R'_2 \vee \ldots \vee R'_n$

$$R'_i = R_i \wedge \bigwedge_{j \neq i} x_j = x'_j$$

$$\begin{aligned}
\exists \vec{x}'.\, R \wedge f(\vec{x}') &\equiv \exists \vec{x}'.\, (R'_1 \wedge f(\vec{x}')) \vee \ldots \vee (R'_n \wedge f(\vec{x}')) \\
&\equiv \big(\exists \vec{x}'.\, R'_1 \wedge f(\vec{x}')\big) \vee \ldots \vee \big(\exists \vec{x}'.\, R'_n \wedge f(\vec{x}')\big)
\end{aligned}$$

$$\begin{aligned}
\exists \vec{x}'.\, R'_i \wedge f(\vec{x}') &\equiv \exists \vec{x}'.\, R_i \wedge \big(\bigwedge_{j \neq i} x_j = x'_j\big) \wedge f(\vec{x}') \\
&\equiv \exists x'_i.\, R_i(\vec{x}, x'_i) \wedge f(x_1, \ldots, x_{i-1}, x'_i, x_{i+1}, \ldots, x_m)
\end{aligned}$$

Synchronous model: $R = R_1 \wedge R_2 \wedge \ldots \wedge R_n$

$$\exists \vec{x}' .\ R_1(\vec{x}, \vec{x}') \wedge \ldots \wedge R_n(\vec{x}, \vec{x}') \wedge f(\vec{x}')$$

- relations R_i are local
- „early” quantification
- heuristics

Example: 3-bit counter

$$
\begin{array}{lcl}
R_0(\vec{x}, x_0') & = & (x_0' = \neg x_0) \\
R_1(\vec{x}, x_1') & = & (x_1' = x_0 \text{ xor } x_1) \\
R_2(\vec{x}, x_2') & = & (x_2' = (x_0 \wedge x_1) \text{ xor } x_2)
\end{array}
$$

$$\exists x_2' \exists x_1' \exists x_0'.\ f(x_0', x_1', x_2') \ \wedge\ R_0(\vec{x}, x_0') \ \wedge\ R_1(\vec{x}, x_1') \ \wedge\ R_2(\vec{x}, x_2')$$

$$\exists x_2' \big(\exists x_1' \exists x_0'.\ f(x_0', x_1', x_2') \ \wedge\ R_0(\vec{x}, x_0') \ \wedge\ R_1(\vec{x}, x_1')\big) \ \wedge\ R_2(\vec{x}, x_2')$$

$$\exists x_2' \big(\exists x_1' \big(\exists x_0'.\ f(x_0', x_1', x_2') \ \wedge\ R_0(\vec{x}, x_0')\big) \ \wedge\ R_1(\vec{x}, x_1')\big) \ \wedge\ R_2(\vec{x}, x_2')$$

$$\exists x_2' \big(\exists x_1' \big(\exists x_0'.\ f(x_0', x_1', x_2') \ \wedge\ R_0(x_0, x_0')\big) \ \wedge\ R_1(x_0, x_1, x_1')\big) \ \wedge\ R_2(x_0, x_1, x_2, x_2')$$

Example: 3-bit counter

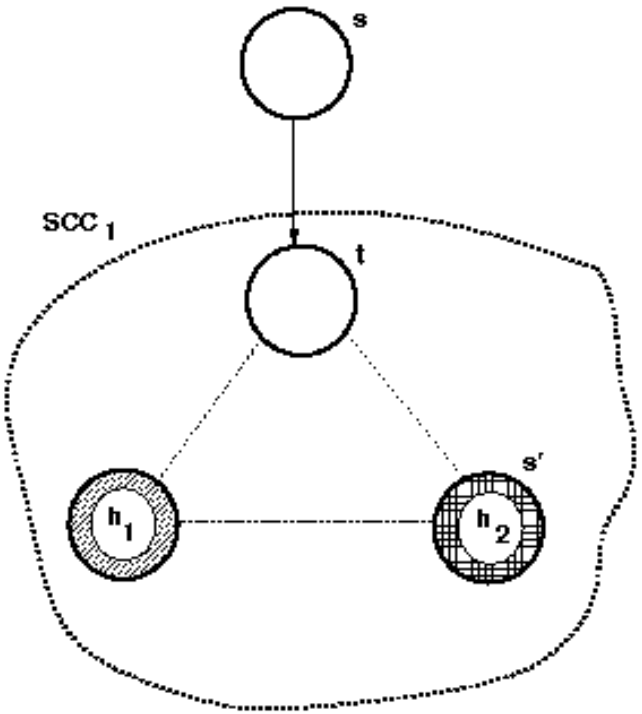
$$\exists x'_2 \quad (\exists x'_1 \quad (\exists x'_0 \quad f(x'_0, x'_1, x'_2) \ \wedge \ R_0(x_0, x'_0)) \\ \wedge \quad R_1(x_0, x_1, x'_1)) \\ \wedge \quad R_2(x_0, x_1, x_2, x'_2)$$

- sequence of $\exists\wedge$ operations
- optimal order of processes (not variables this time):
  - early elimination of variables ($\exists$)
  - late introducing of variables

(Counter)examples

Counterexample

counterexample for $\mathrm{AF}\,\phi$ = example for $\mathrm{EG}\,\neg\phi$



[Clarke, Grumberg, Long 1994]

Counterexample

counterexample for $\mathrm{AF}\,\phi$ = example for $\mathrm{EG}\,\neg\phi$

counterexample for $\mathrm{AG}\,\phi$ = example for $\mathrm{EF}\,\neg\phi$

(fair counterexample is always an infinite path)

Counterexample

counterexample for $\mathrm{AF}\,\phi$ = example for $\mathrm{EG}\,\neg\phi$

counterexample for $\mathrm{AG}\,\phi$ = example for $\mathrm{EF}\,\neg\phi$

(fair counterexample is always an infinite path)

counterexample for $\mathrm{EF}\,\phi$ = **?**

counterexample for $\mathrm{EG}\,\phi$ = **?**

Symbolic counterexample

How to compute an example for:

- EG ϕ
- E ϕ U ψ
- EX ϕ

symbolically ?

Example for E_U_

Computation of E f U g:

$$Q_0 \subseteq Q_1 \subseteq \ldots \quad (1 \leq i \leq n)$$

$s \in Q_j \iff g$ may be reached from s "via f" by $\leq j$ transitions

Computing an example for $s \models$ E f U g:

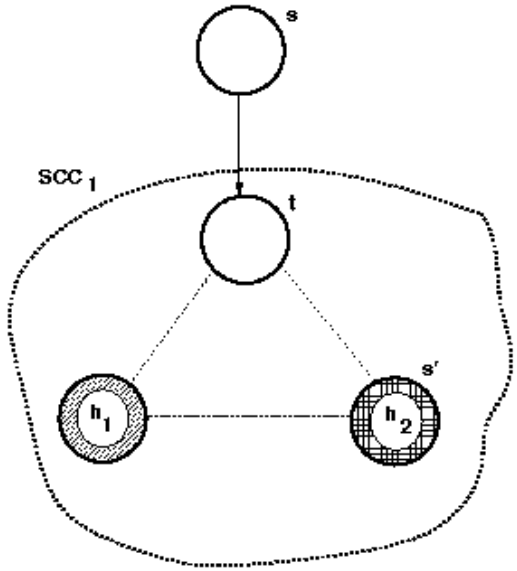
- let j minimal s.t. $s \in Q_j$
- reconstruct $s = s_j \rightarrow s_{j-1} \rightarrow \ldots \rightarrow s_0 \in g$

Symbolic counterexample

How to compute a fair example for:

- EG ϕ
- E ϕ U ψ
- EX ϕ

symbolically ?



[Clarke, Grumberg, Long 1994]

Fair example for EG _

$$\text{EG}\, f \;=\; \nu Z.\; f \;\wedge\; \bigwedge_{i=1}^{n} \text{EX E}\, f\, \text{U}\, (h_i \wedge Z)$$

last iteration $Z \;\mapsto\; f \;\wedge\; \bigwedge_{i=1}^{n} \text{EX E}\, f\, \text{U}\, (h_i \wedge Z)$:

computation of $\text{E}\, f\, \text{U}\, (h_i \wedge Z)$: $\qquad Z = \text{EG}\, f$

$$Q_0^i \;\subseteq\; Q_1^i \;\subseteq\; \ldots \qquad (1 \leq i \leq n)$$

$s \in Q_j^i \iff (h_i \wedge \text{EG}\, f)$ may be reached from s "via f"

by $\leq j$ transitions

Fair example for EG _

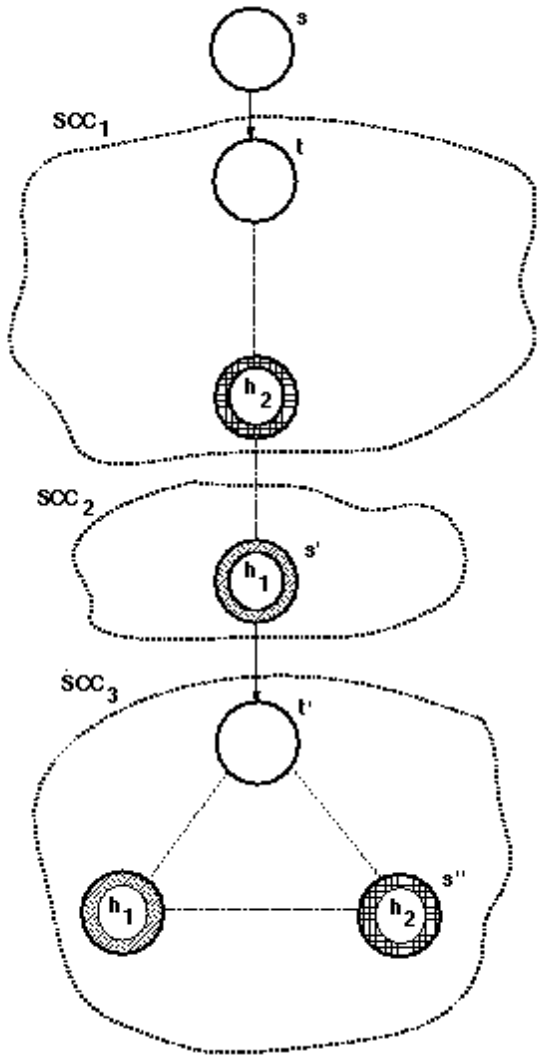
$$\mathsf{EG}\, f \;=\; \nu Z.\; f \;\wedge\; \bigwedge_{i=1}^{n} \mathsf{EX}\, \mathsf{E}\, f\, \mathsf{U}\, (h_i \wedge Z)$$

$s := s_0$ initial state
$I := \{1, \ldots, n\}$
repeat
  find t s.t. $s \to t$, $t \in Q_j^i$, $i \in I$, j minimal
  reconstruct $t = t_j \to t_{j-1} \to \ldots \to t_0 \in (h_i \wedge \mathsf{EG}\, f)$
  $I := I \setminus \{i \mid t_0 \in h_i\}$
  $s := t_0$
until $I = \emptyset$
$s' := s$

$I := I \setminus \{i \mid t \in Q_j^i\}$

$\mapsto$ path $s_0 \to \ldots \to s'$

Fair example for EG _



[Clarke, Grumberg, Long 1994]

Fair example for EG _

$$\mathsf{EG}\, f \;=\; \nu Z.\; f \;\wedge\; \bigwedge_{i=1}^{n} \mathsf{EX}\, \mathsf{E}\, f\, \mathsf{U}\, (h_i \wedge Z)$$

we have a path $s_0 \to \ldots \to s'$ let $t =$ the first t_0

(a) if $s' \vDash \mathsf{EX}\, \mathsf{E}\, f\, \mathsf{U}\, \{t\}$ stop

otherwise restart: $s_0 := s', \quad I := \{1, \ldots, n\}$

improvement:

(b) compute $\mathsf{E}\, f\, \mathsf{U}\, \{t\}$

as long as $\neg(s \vDash \mathsf{E}\, f\, \mathsf{U}\, \{t\})$, restart: $s_0 := s$, $I := \{1, \ldots, n\}$

Fair example for E_U_, EX_

Example for $\mathsf{E}\,\phi\,\mathsf{U}\,(\psi \wedge \texttt{fair})$ or $\mathsf{EX}\,(\phi \wedge \texttt{fair})$ extend with a fair example for $\mathsf{EG}\,\texttt{true}$.

What else can be computed using OBDDs ?

- $L_\omega(\mathcal{A}) \neq \emptyset$ — fair EG $\mathtt{true}$
- LTL model checking
- $L_\omega(\mathcal{A}_1) \subseteq L_\omega(\mathcal{A}_2)$ — $\mathcal{A}_1 \times \mathcal{A}_2 \models \mathsf{A}(\mathsf{G}\,\mathsf{F}\,q_1 \implies \mathsf{G}\,\mathsf{F}\,q_2)$
- μ-calculus model checking
- reachable states
- deadlocks
- (bi)simulation equivalence
- ...

What else can be computed using OBDDs ?

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OBDDs are routinely used in hardware industry nowadays