Which extensions of vector addition systems have decidable reachability ?

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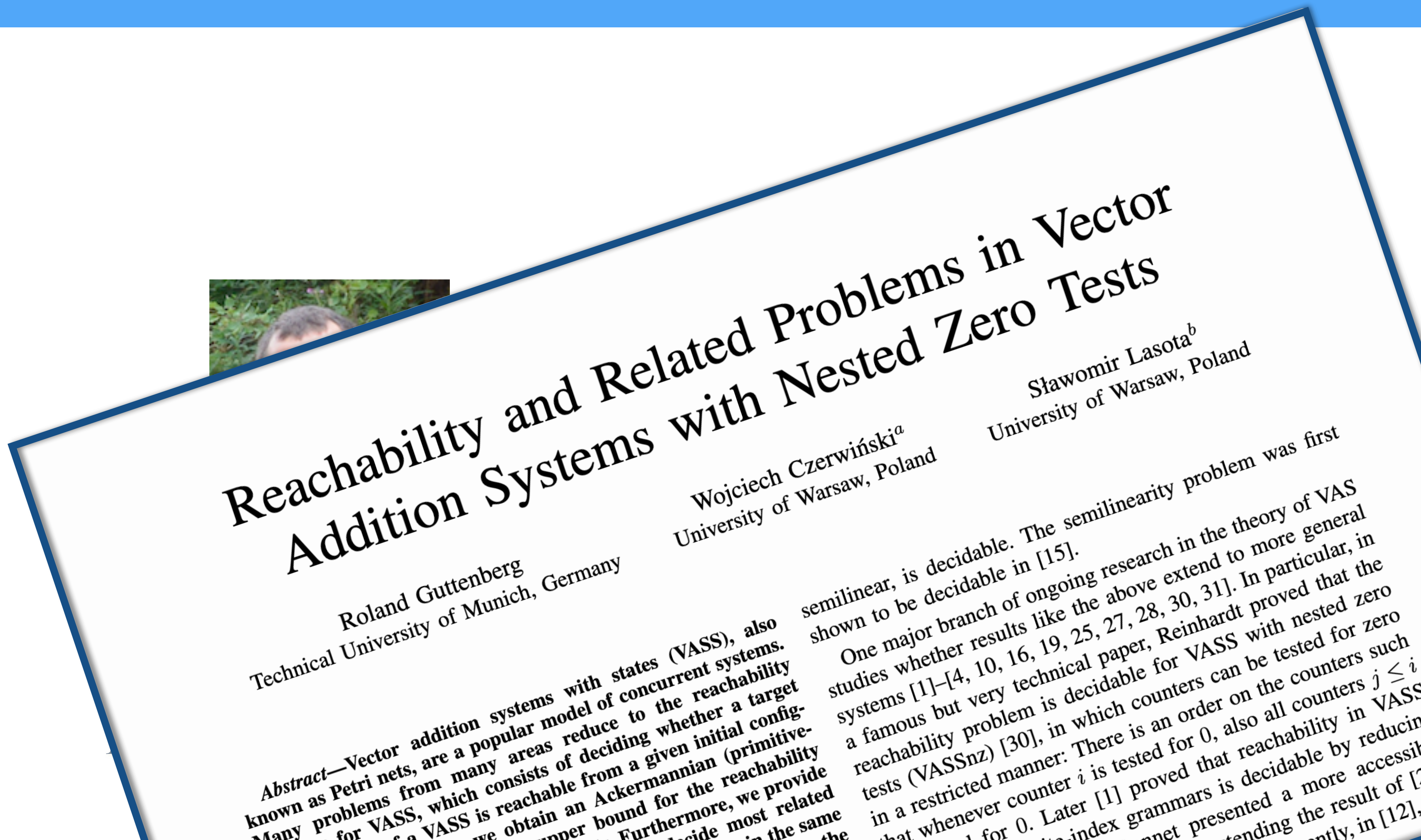

Reachability and Related Problems in Vector Addition Systems with Nested Zero Tests

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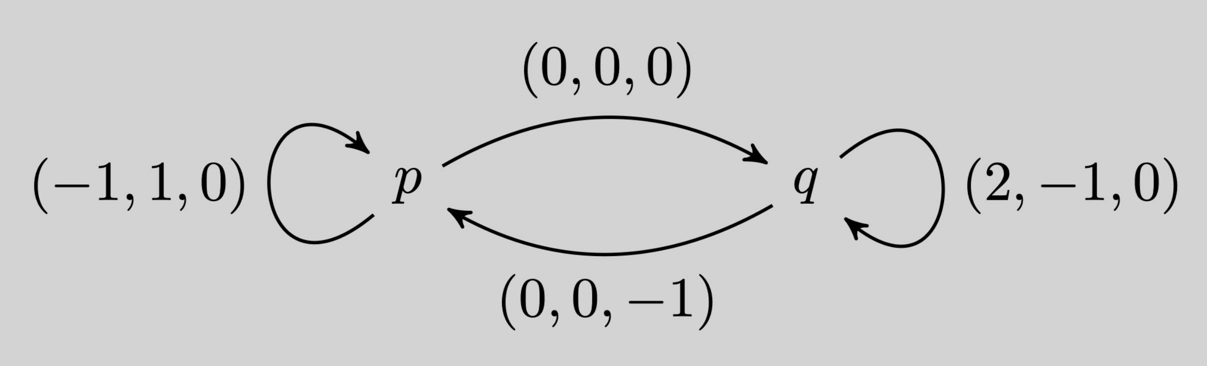
Abstract—Vector addition systems with states (VASS), also known as Petri nets, are a popular model of concurrent systems. Many problems from many areas reduce to the reachability ... for VASS, which consists of deciding whether a target ... of a VASS is reachable from a given initial config- ... we obtain an Ackermannian (primitive- ... upper bound for the reachability ... Furthermore, we provide ... decide most related ... in the same ...

semilinear, is decidable. The semilinearity problem was first shown to be decidable in [15].

One major branch of ongoing research in the theory of VAS studies whether results like the above extend to more general systems [1]–[4, 10, 16, 19, 25, 27, 28, 30, 31]. In particular, in a famous but very technical paper, Reinhardt proved that the reachability problem is decidable for VASS with nested zero tests (VASSnz) [30], in which counters can be tested for zero in a restricted manner: There is an order on the counters such that whenever counter i is tested for 0, also all counters $j \leq i$ ...

VASS relationally



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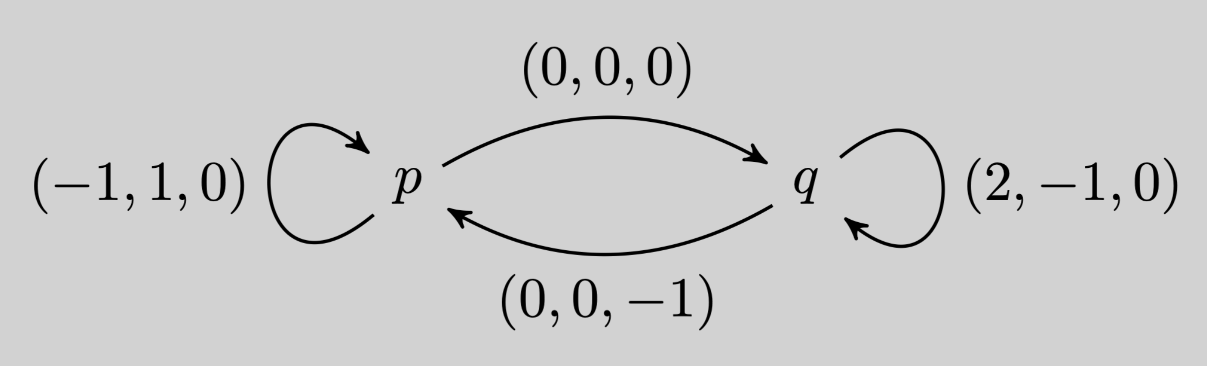

$$\mathrm{VA}_{(2,-1,0)} = \{(x, y) \mid y = x + (2, -1, 0)\} \subseteq \mathbb{N}^3 \times \mathbb{N}^3$$

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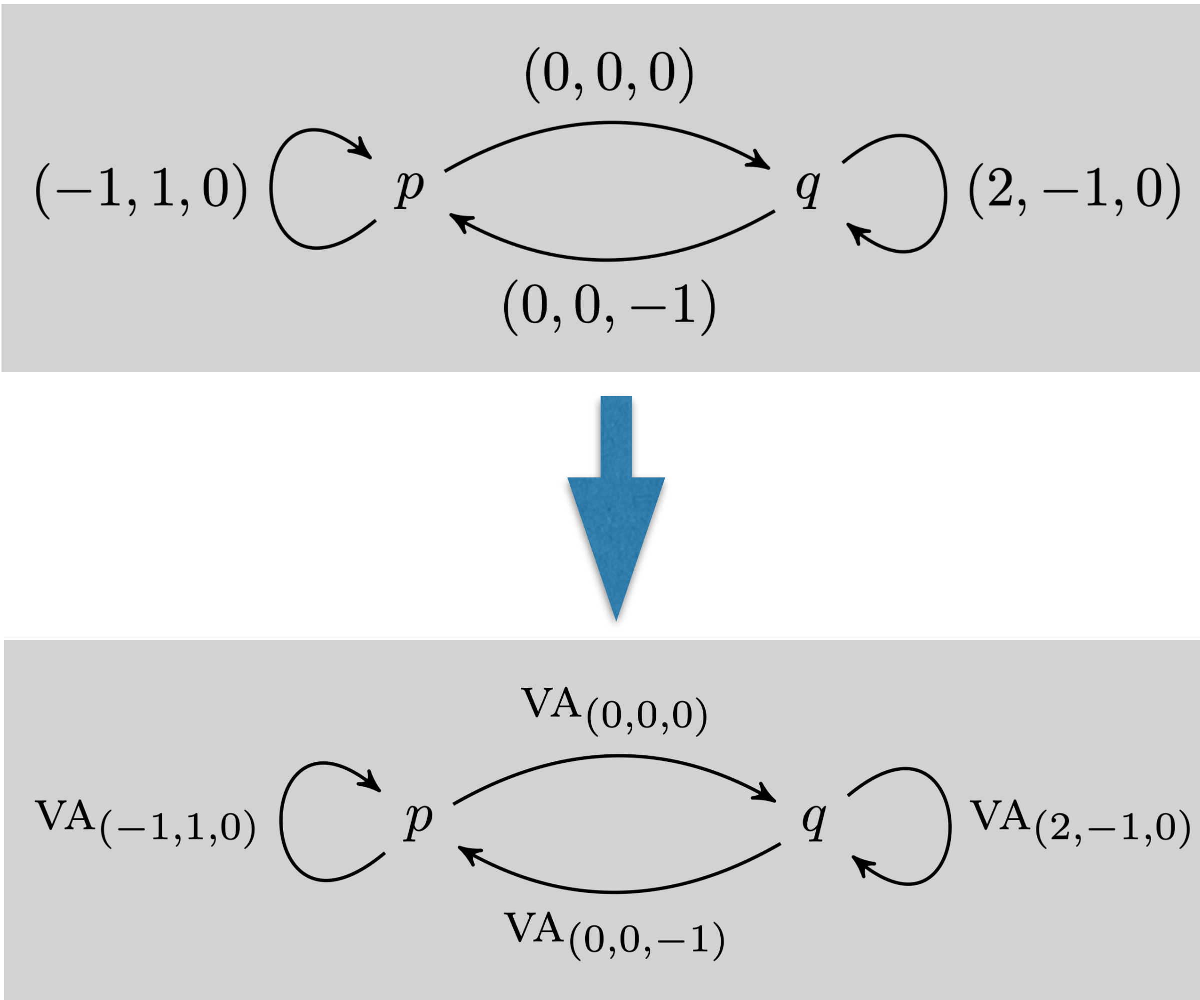

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Vector Addition relations

$$\mathrm{VA} = \{\mathrm{VA}_v \mid v \in \mathbb{N}^d\}$$

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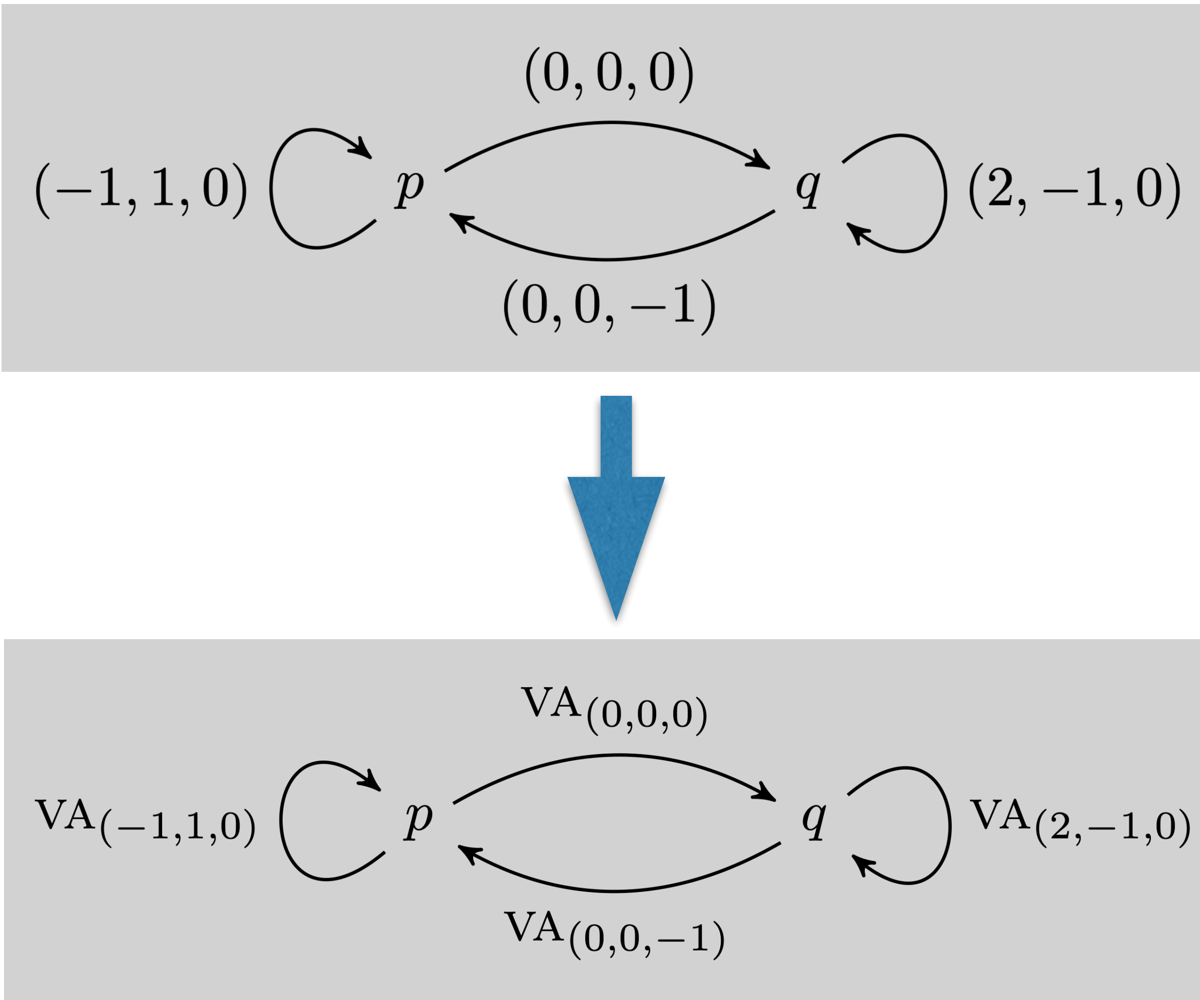

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VASS = VA-systems (with states)

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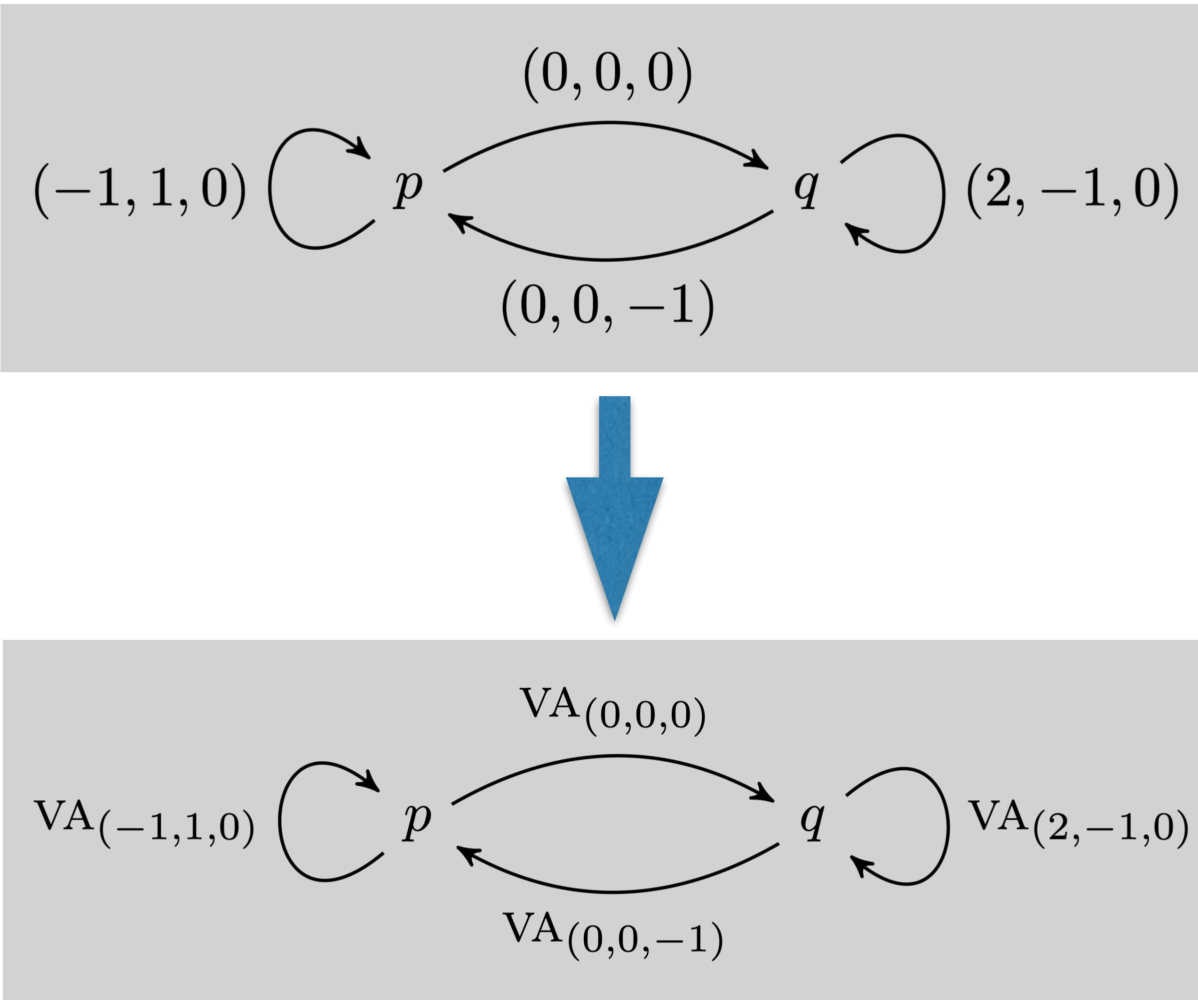

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? = VM-systems

C-systems

$\mathrm{VA} \subseteq \mathcal{C} \subseteq \mathbb{N}^d \times \mathbb{N}^d$ step relations

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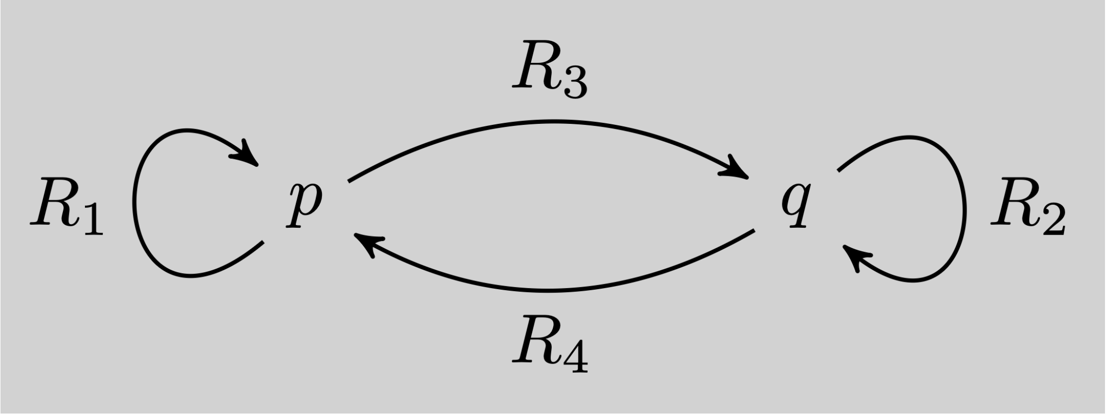

$R_1, R_2, R_3, R_4 \in \mathcal{C}$

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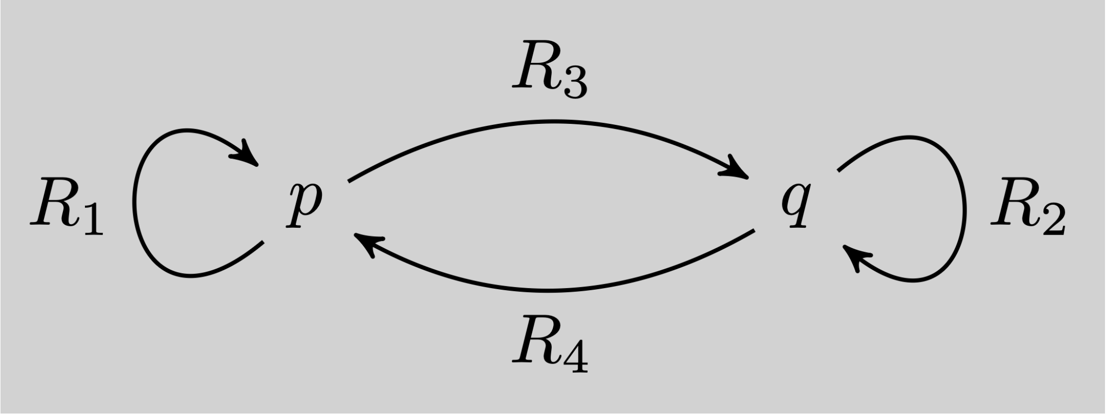

$R_1, R_2, R_3, R_4 \in \mathcal{C}$

admitting finite presentations

Question: For which classes C,
C-systems have decidable reachability?

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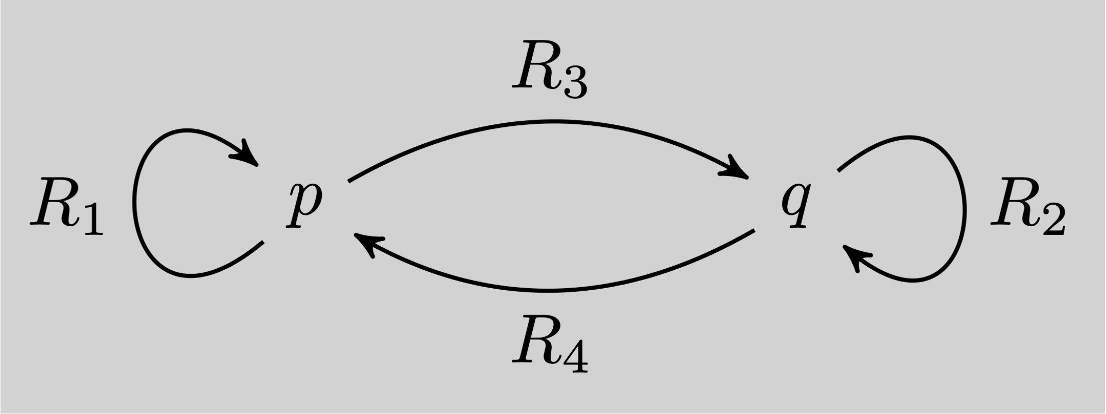

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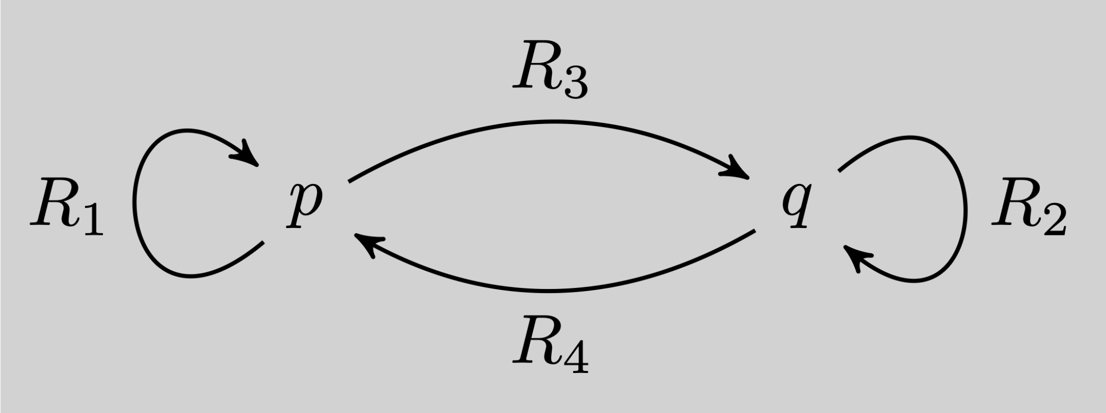

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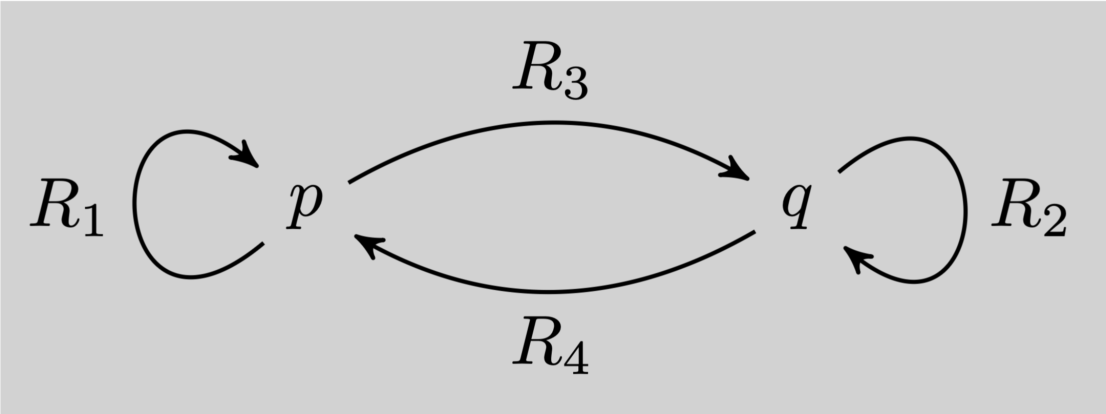

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C = Semilinear ?

Semilinear includes VA ∪ ZT !

$\Longrightarrow$ Semilinear-systems include counter machines !

monotone $\mathcal{C}$-systems

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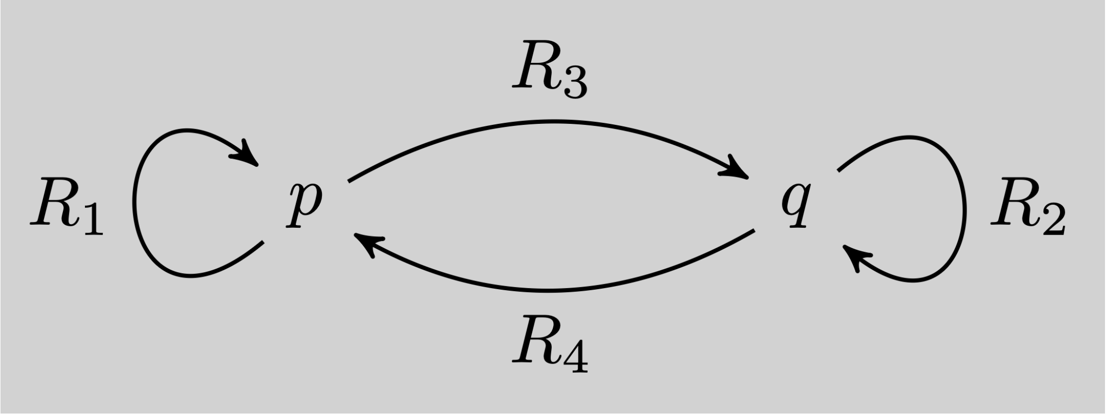

$R_1, R_2, R_3, R_4 \in \mathcal{C}$
and **monotone**

$(x, y) \in R,\ \ v \in \mathbb{N}^d$
$\Downarrow$
$(x+v, y+v) \in R$

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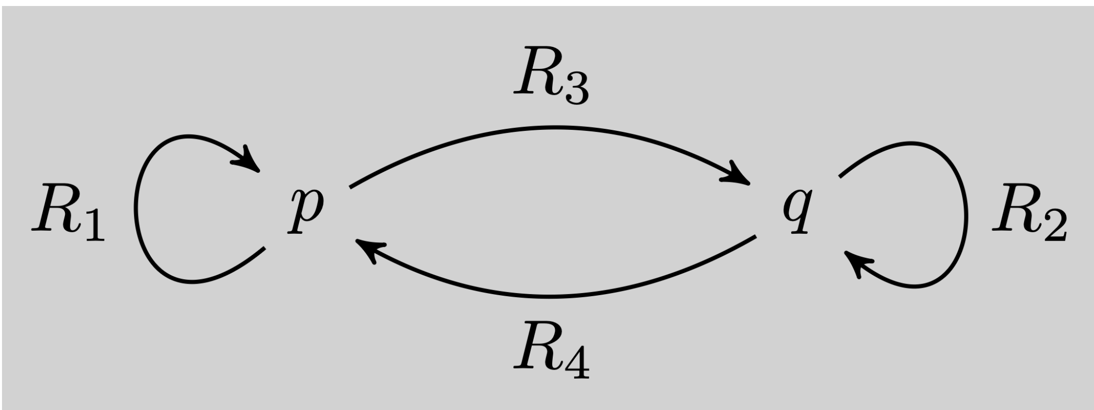

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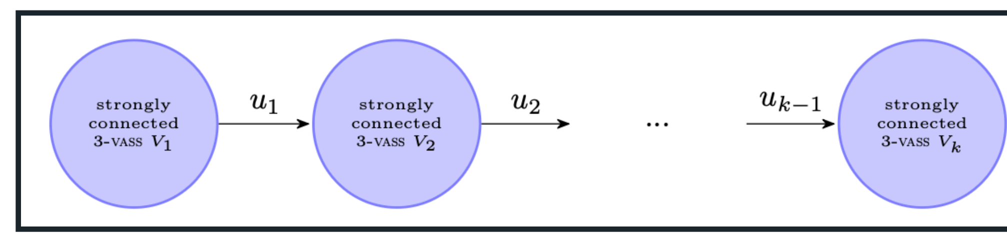

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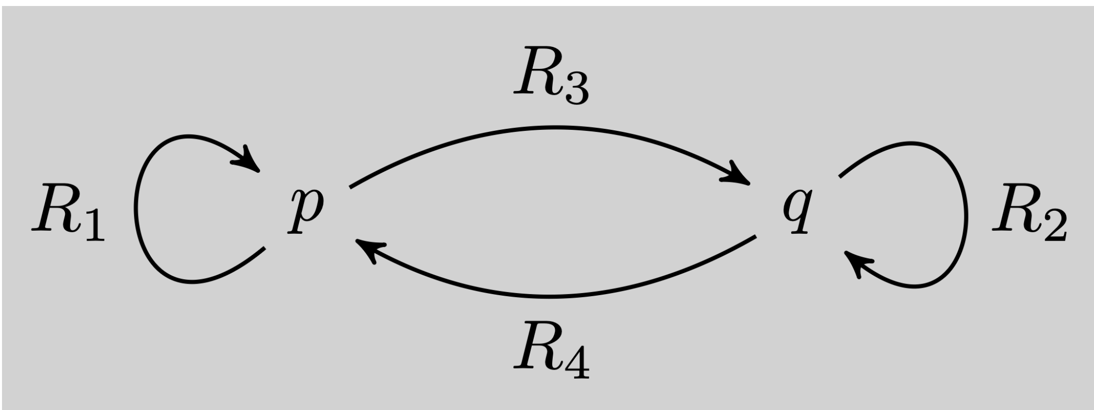

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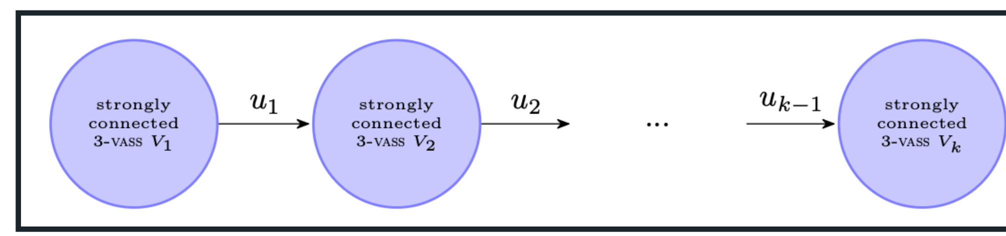

Fact: monotone Semilinear-systems = VASS

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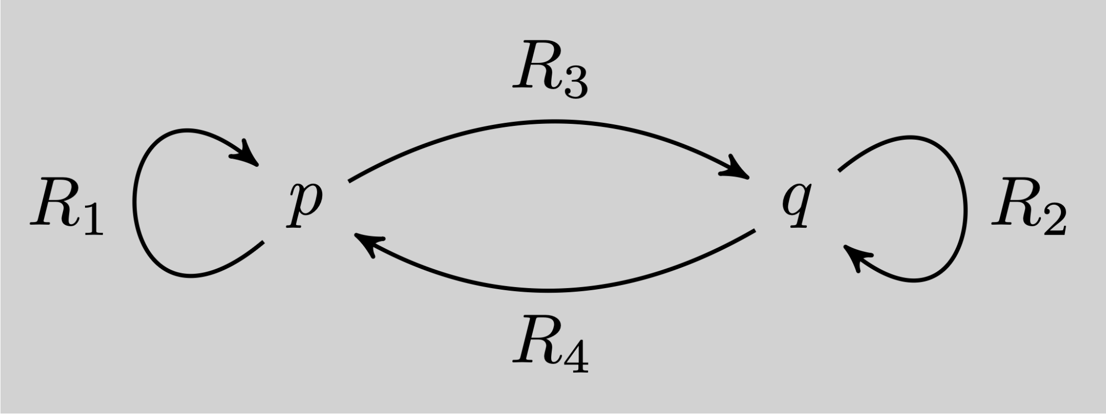

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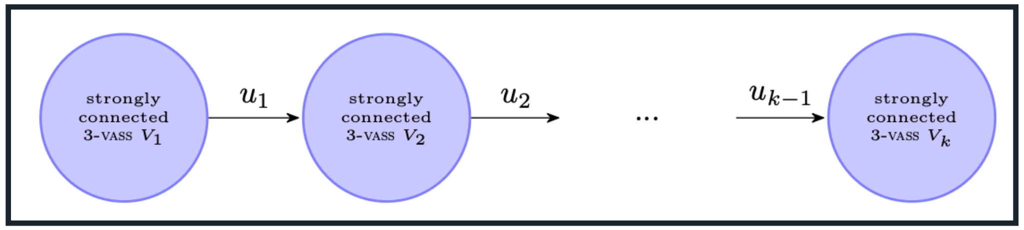

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characterisation

$\mathrm{VA} \subseteq \mathcal{C} \subseteq \mathbb{N}^d \times \mathbb{N}^d$

$C \longmapsto$ sections of **monotone** C-systems

relations obtainable as follows:

1. start with the reachability relation
2. fix some input/output coordinates
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$(k{+}1)$-nested VASS $\longmapsto$ **monotone** (sections of k-nested VASS)-systems

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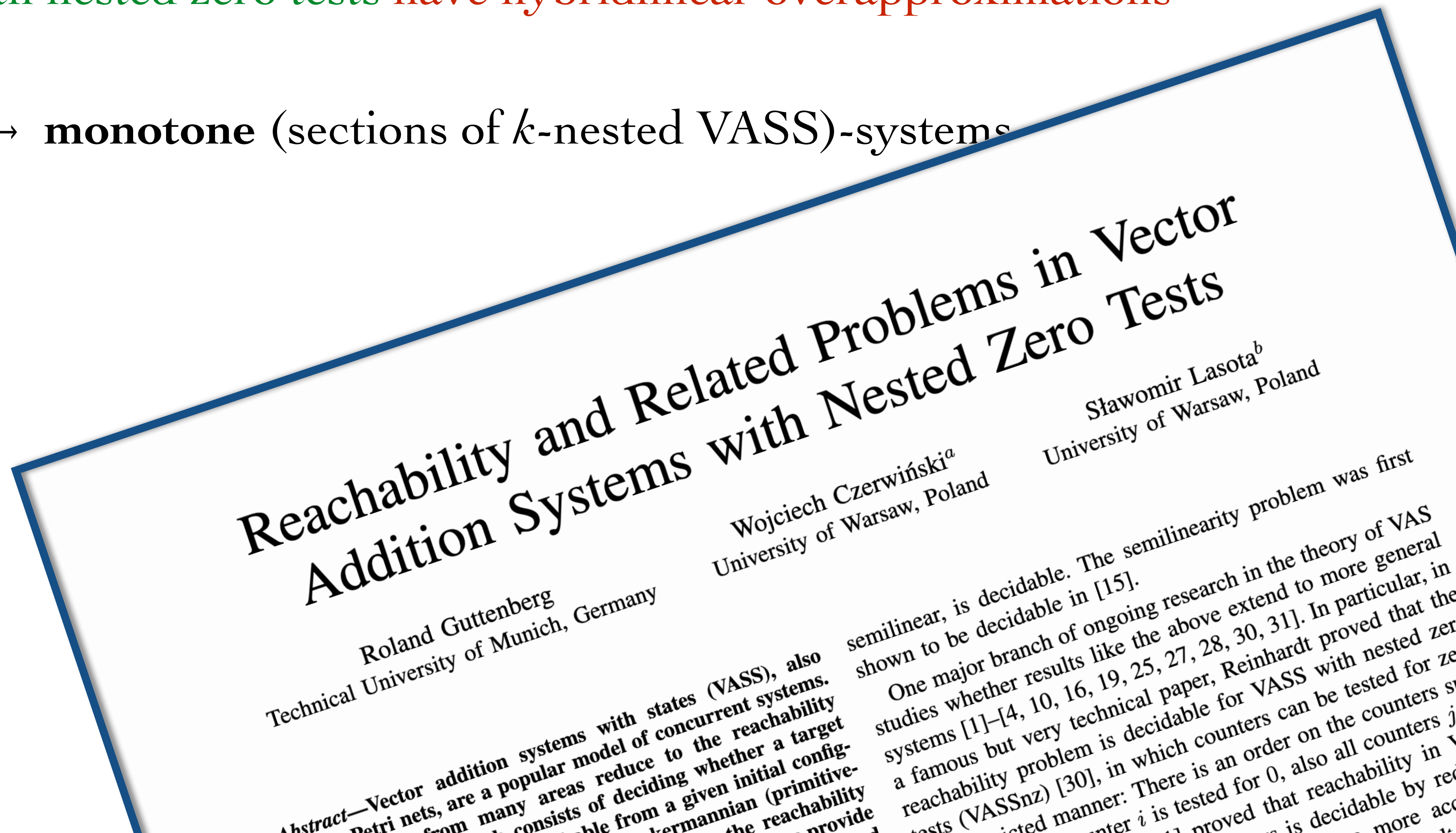
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