Some recent advances in register automata

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I. Introduction to register automata

II. Some recent advances

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Register automata

- finite-memory automata Francez, Kaminski 1990
- history-dependent automata Pistore 1999
- **register automata** Neven, Schwentick, Vianu 2004
- nominal automata Bojańczyk, Klin, L. 2011
- automata with atoms Bojańczyk, Klin, L., Toruńczyk 2013
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- FO over data words
- rigidly guarded MSO
- freeze LTL
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Register automata

- alphabet = $\mathbb{A}$
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- I, F $\subseteq$ Q subsets od initial and accepting states
- $\delta \subseteq Q \times \mathbb{A} \times Q$ transitions

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valuations of n registers storing data values

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transitions compare input to registers and update registers

Register automata - transitions

- **syntactic** presentation:

- **semantic** presentation:

Register automata - transitions

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finite set of rules of the form (c, φ, c'),

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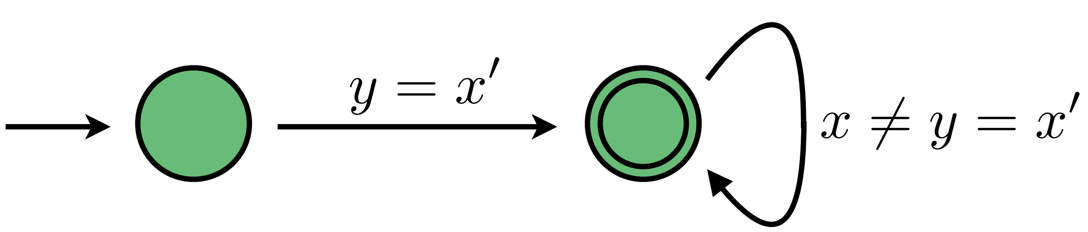

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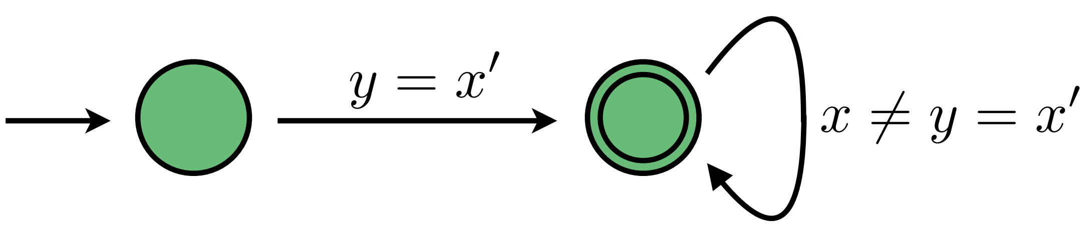

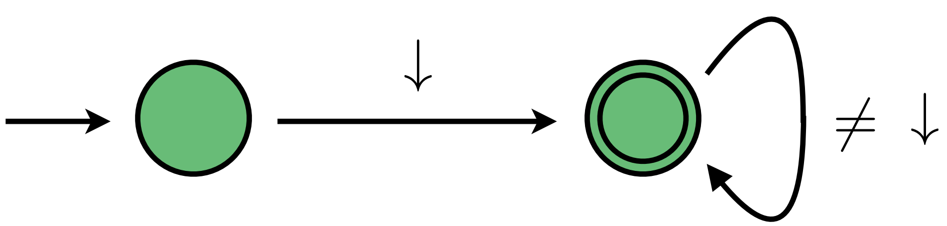

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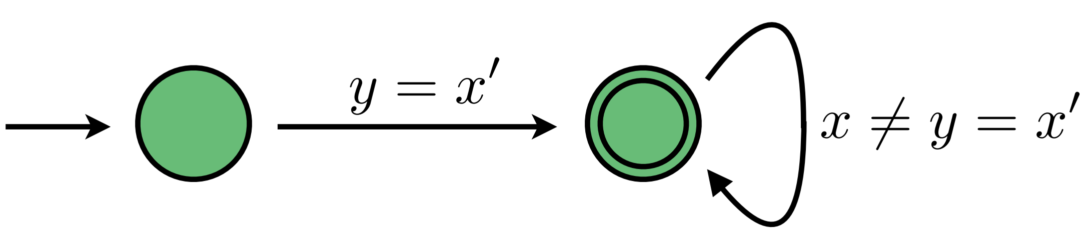

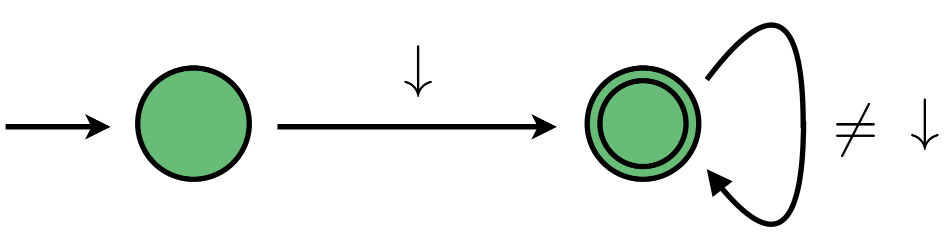

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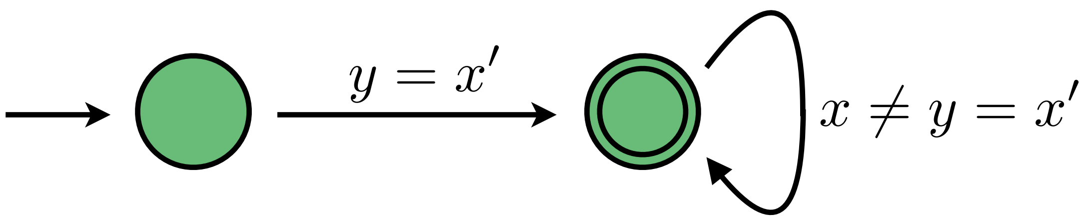

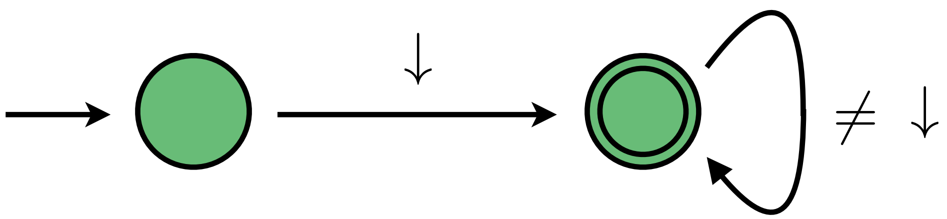

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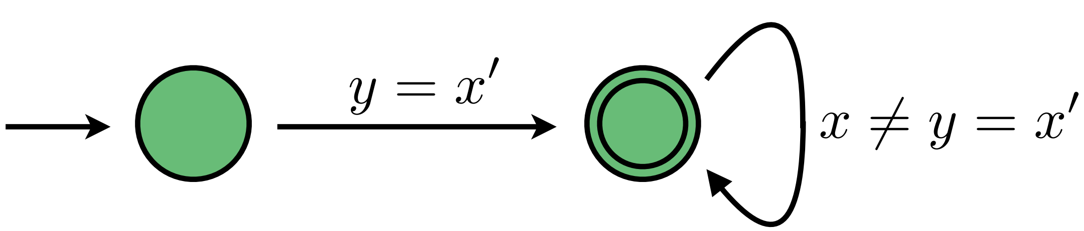

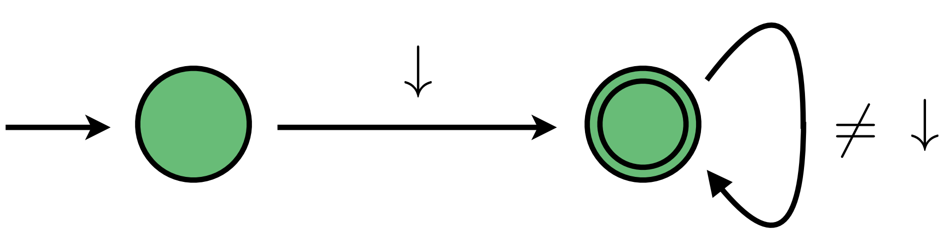

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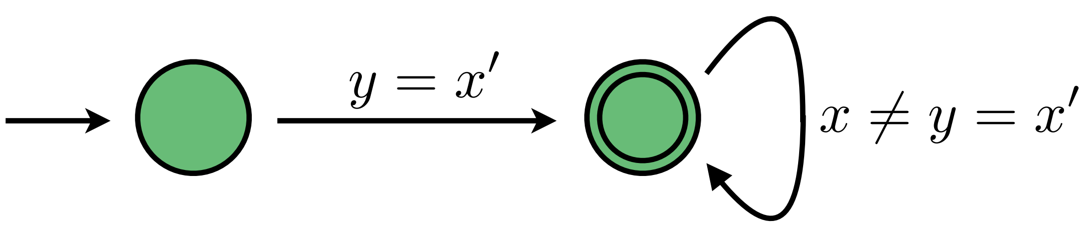

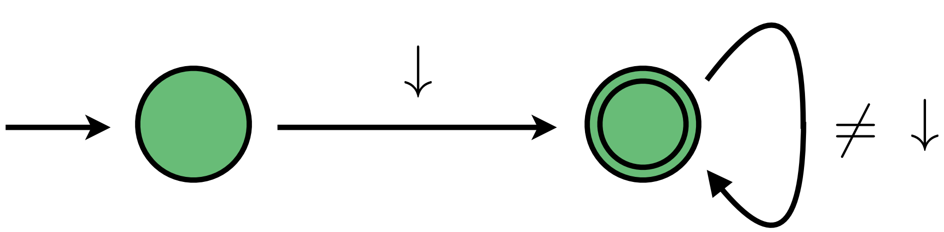

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the presentations are equivalent

Nondeterminism in register automata

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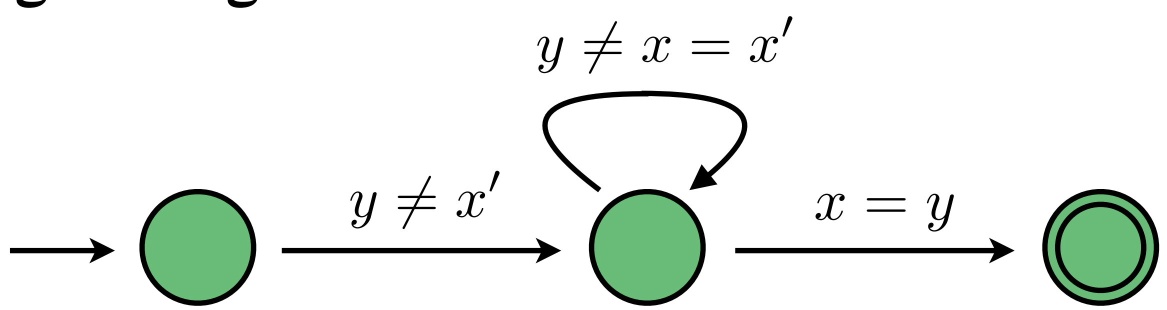

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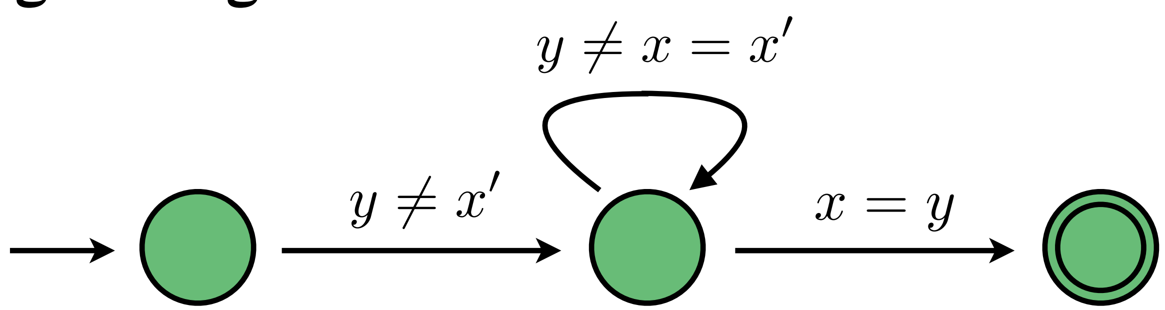

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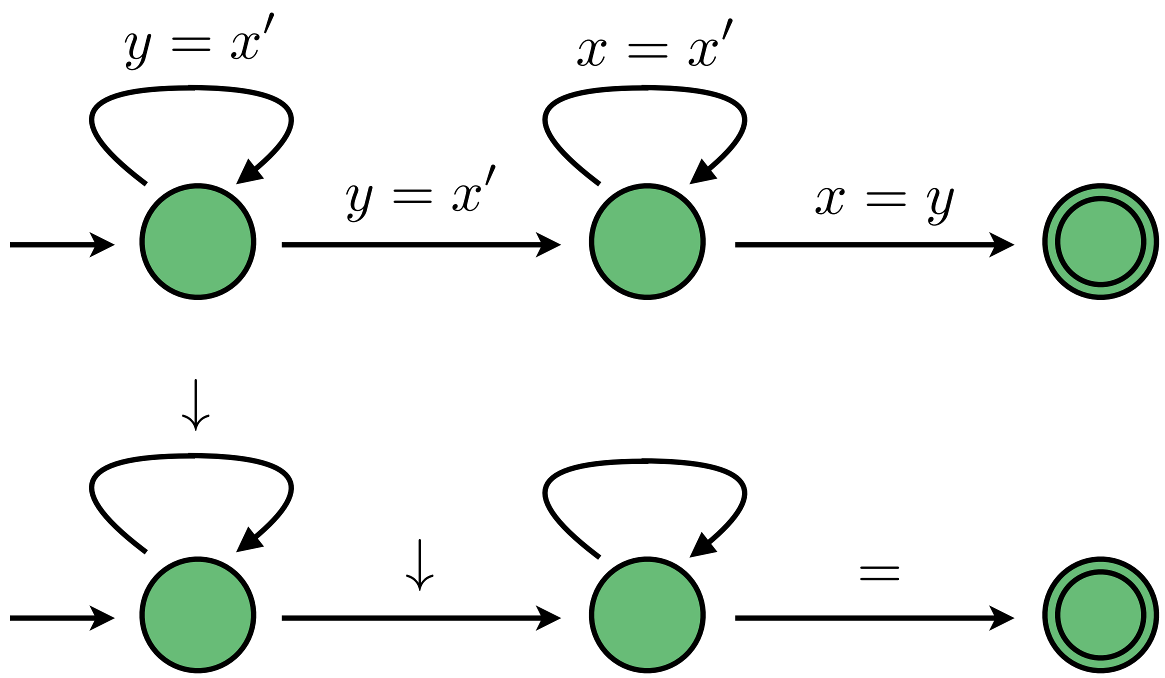

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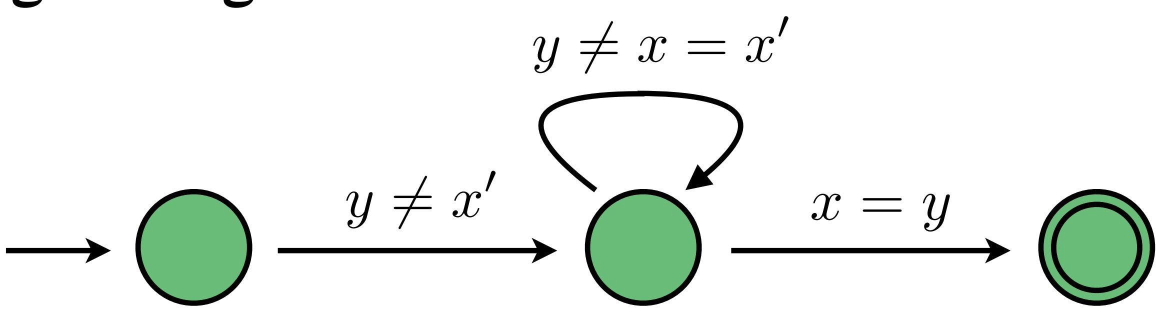

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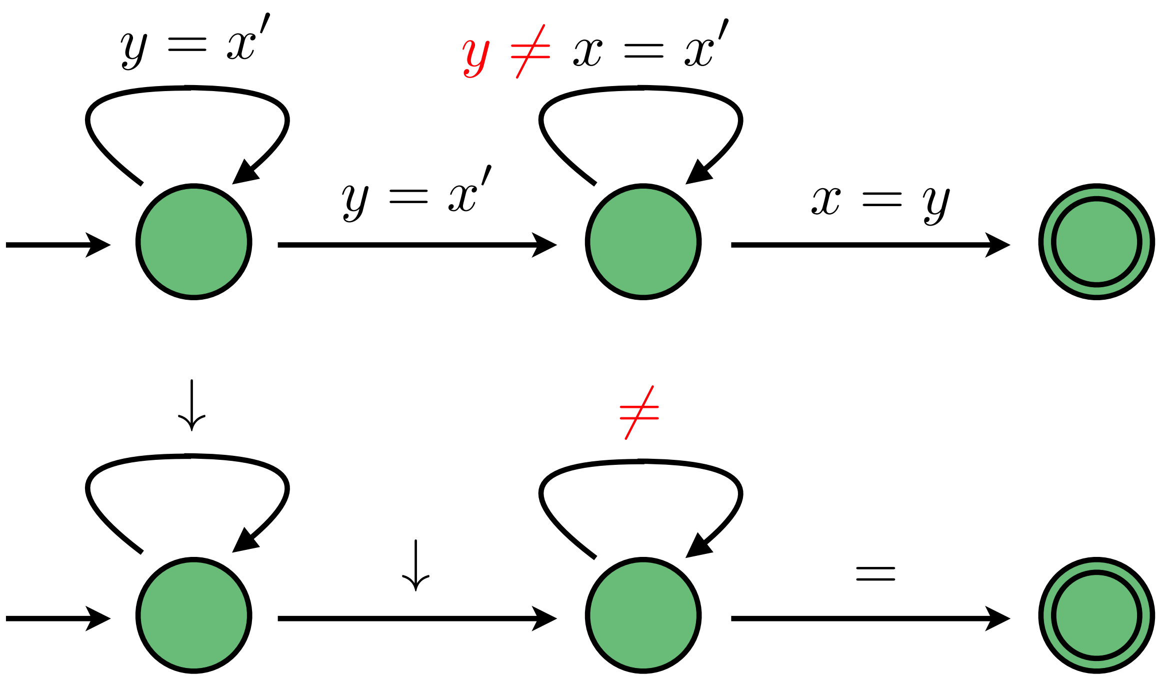

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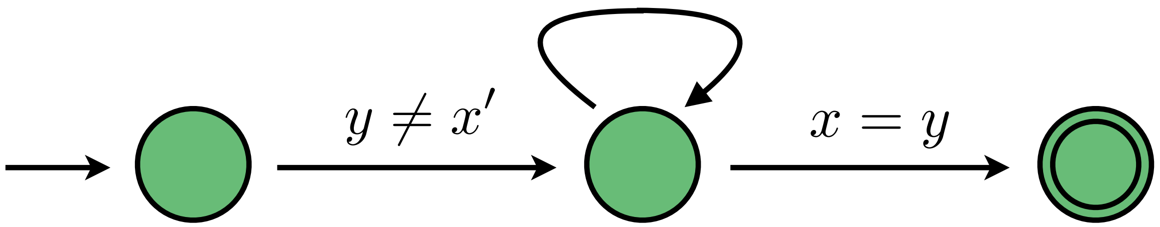

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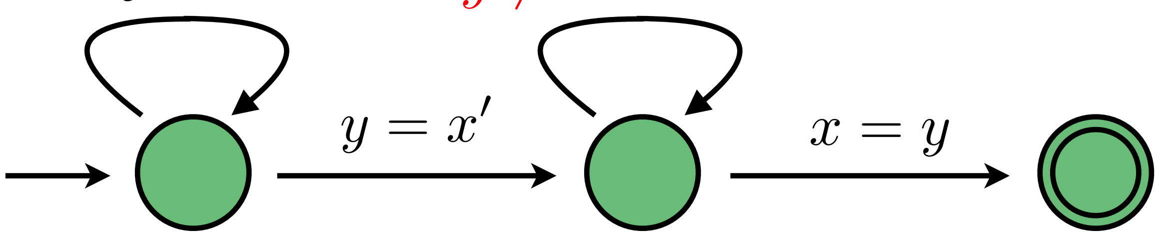

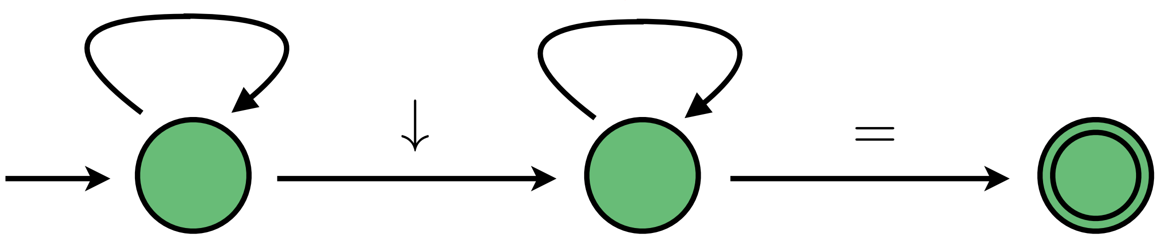

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DRA $\subseteq$ URA $\subseteq$ NRA $\subseteq$ ARA

NRA $\supseteq$ NRA without guessing

ARA $\supseteq$ ARA without guessing

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X and Y are **in the same orbit** if $X = \pi(Y)$ for some data permutation π

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- finite words $\mathbb{A}^*$ — ∞

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- Nondeterministic **orbit-finite automata NOFA**
  - alphabet = $\mathbb{A}$
  - Q - an **orbit-finite** invariant set of states
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- Richer structure of data values than ($\mathbb{A}$, =)

e.g. ($\mathbb{Q}$, <)
any homogeneous/oligomorphic one

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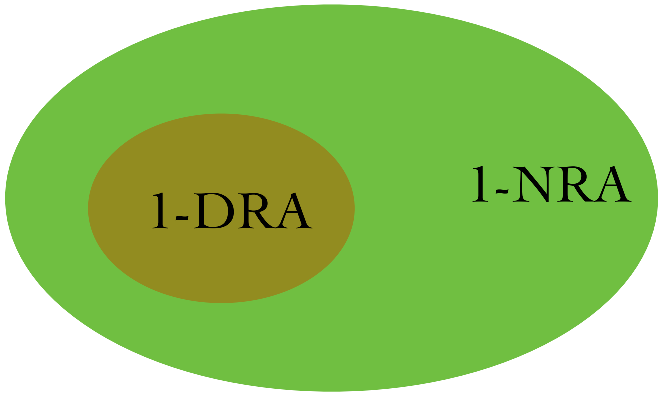

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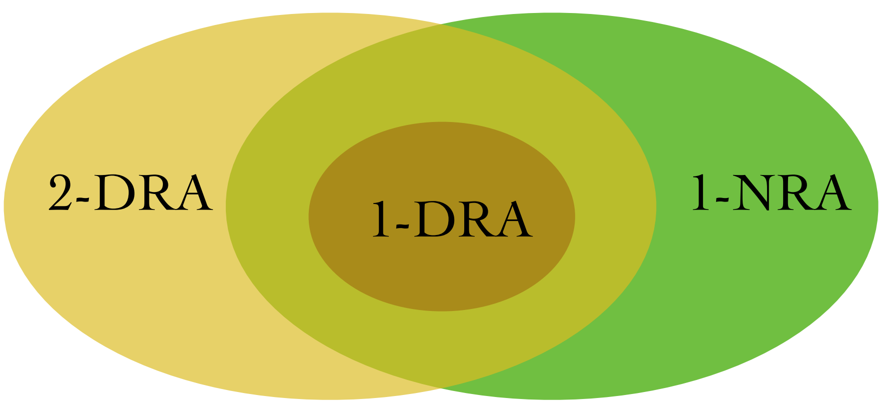

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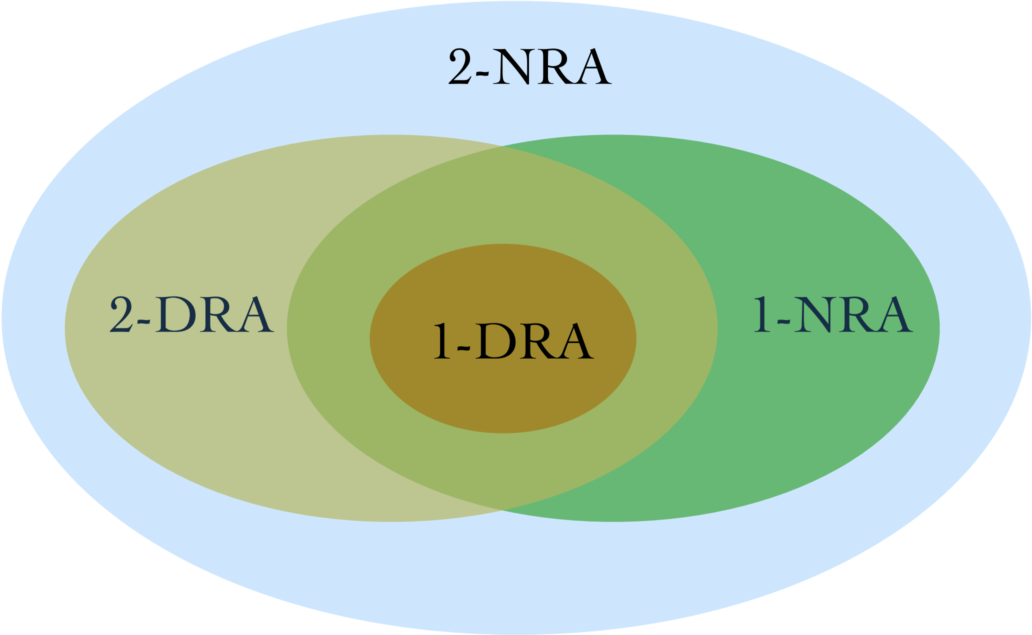

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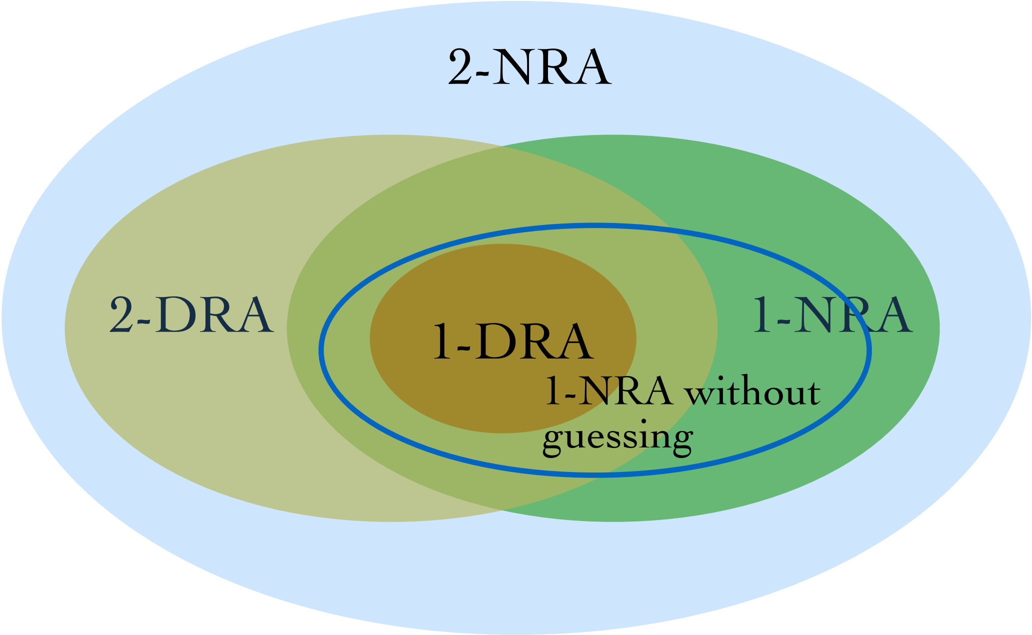

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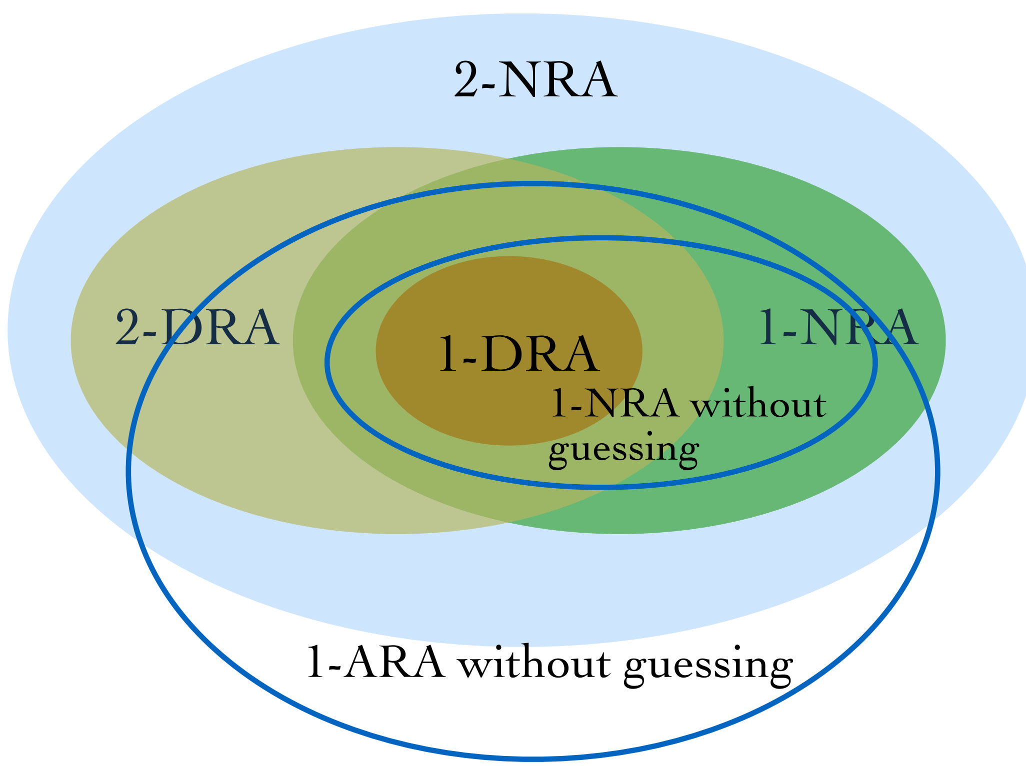

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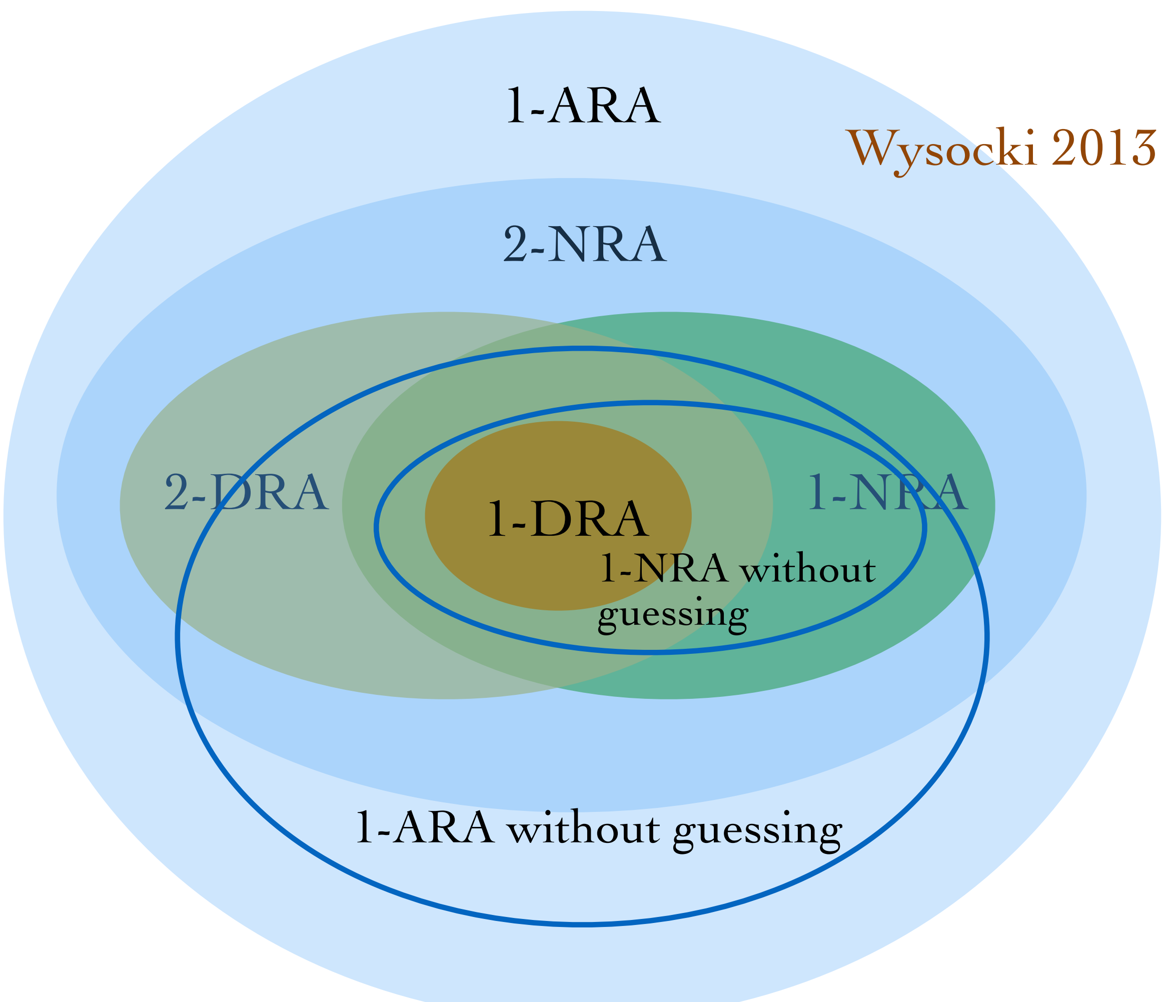

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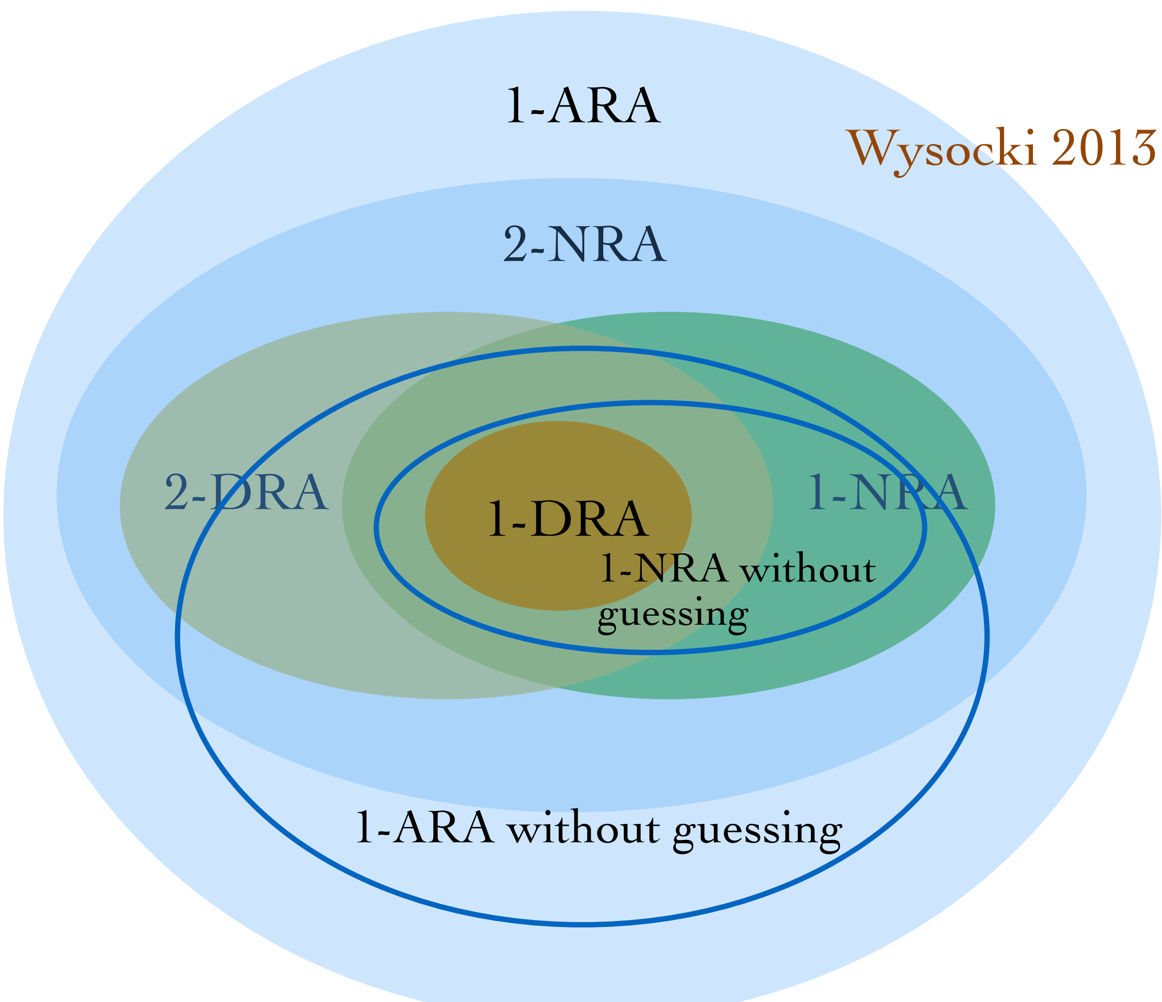
DRA = left congruence of orbit-finite index

Bojańczyk, Klin, L. 2014



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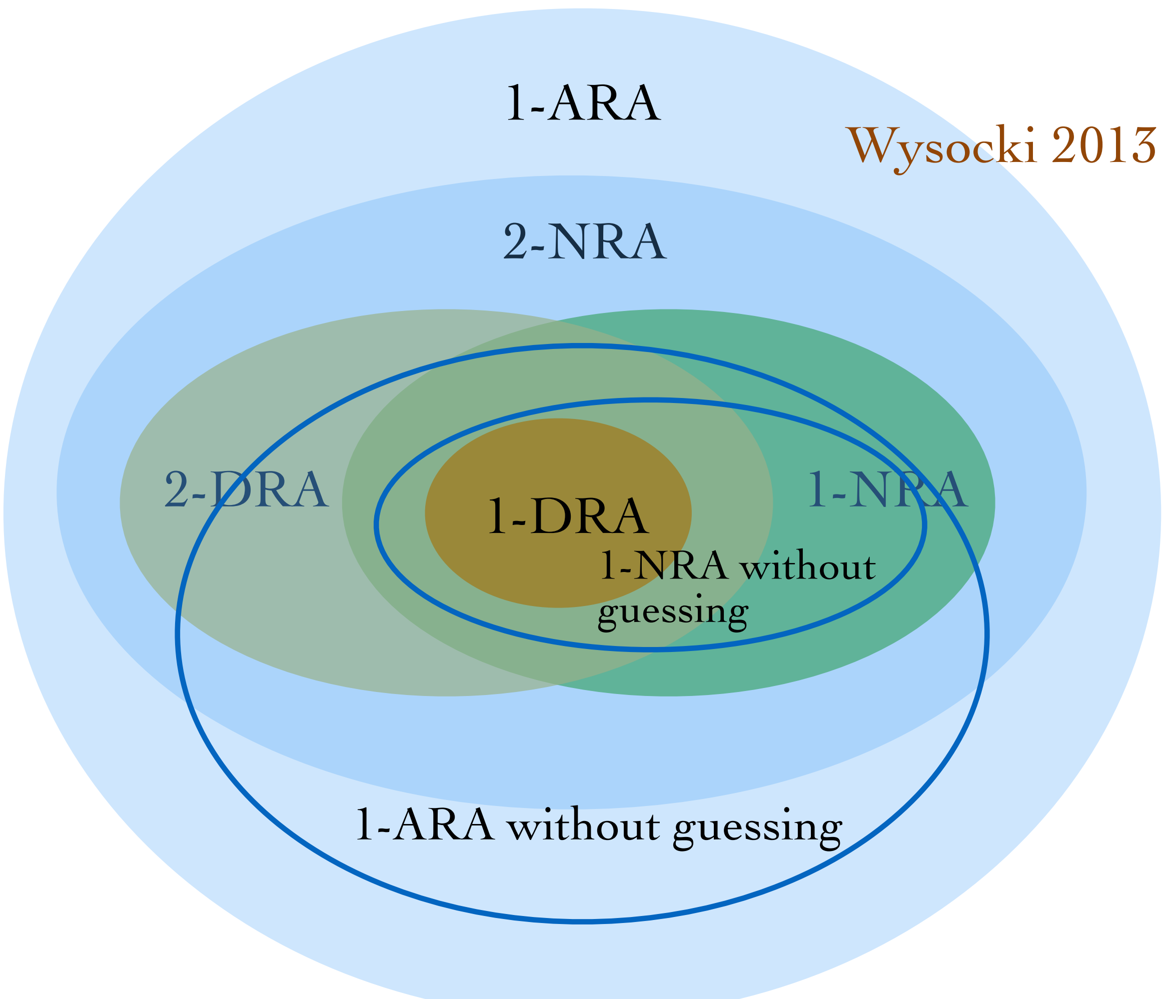

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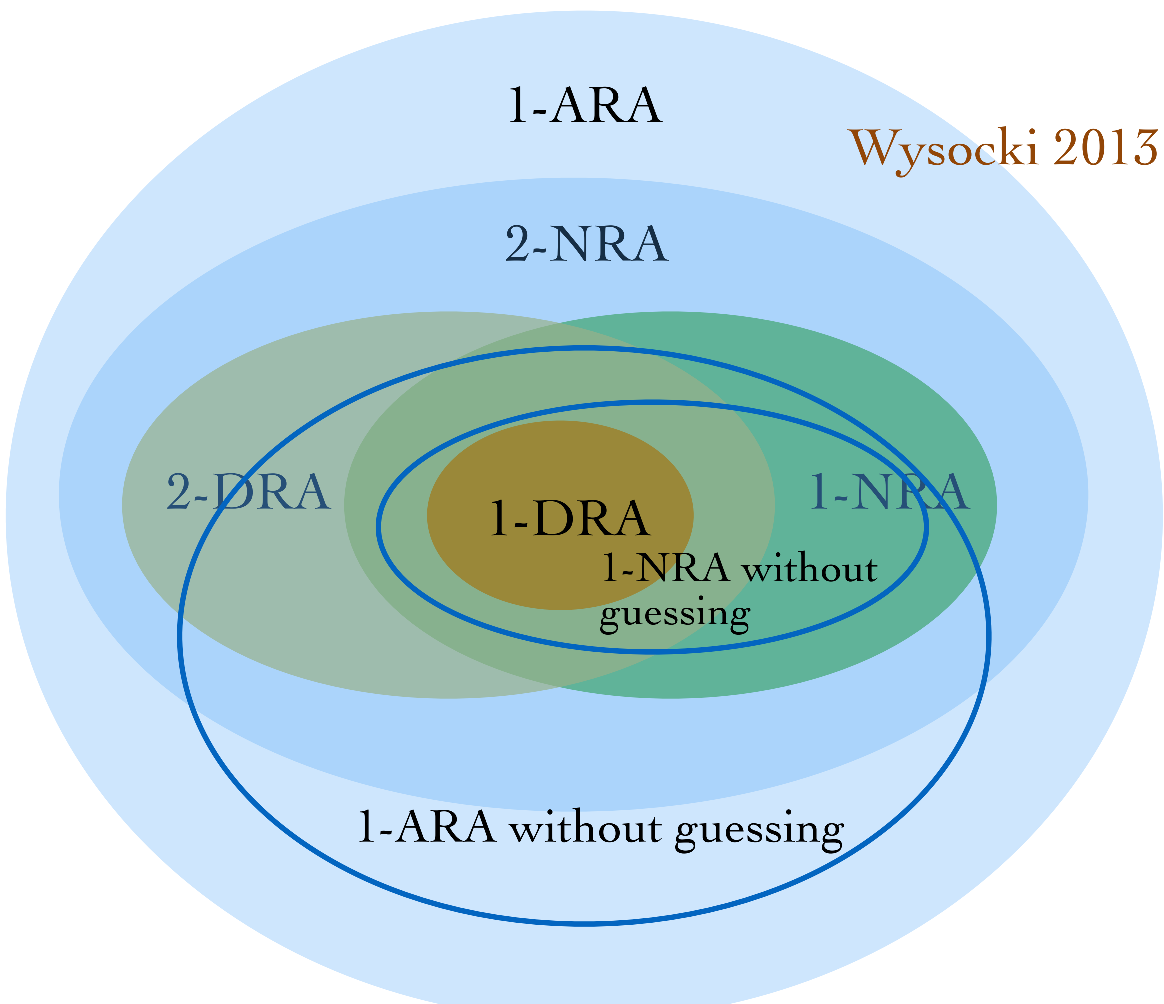
rational expressions?



Rational (regular) expressions

- **unification based** rational expressions Kaminski, Tan 2006 only =
- rational expressions **with binders** Kurz, Suzuki, Tuosto 2012
- rational expressions **with memory** Libkin, Tan, Vrgoc 2015 φ
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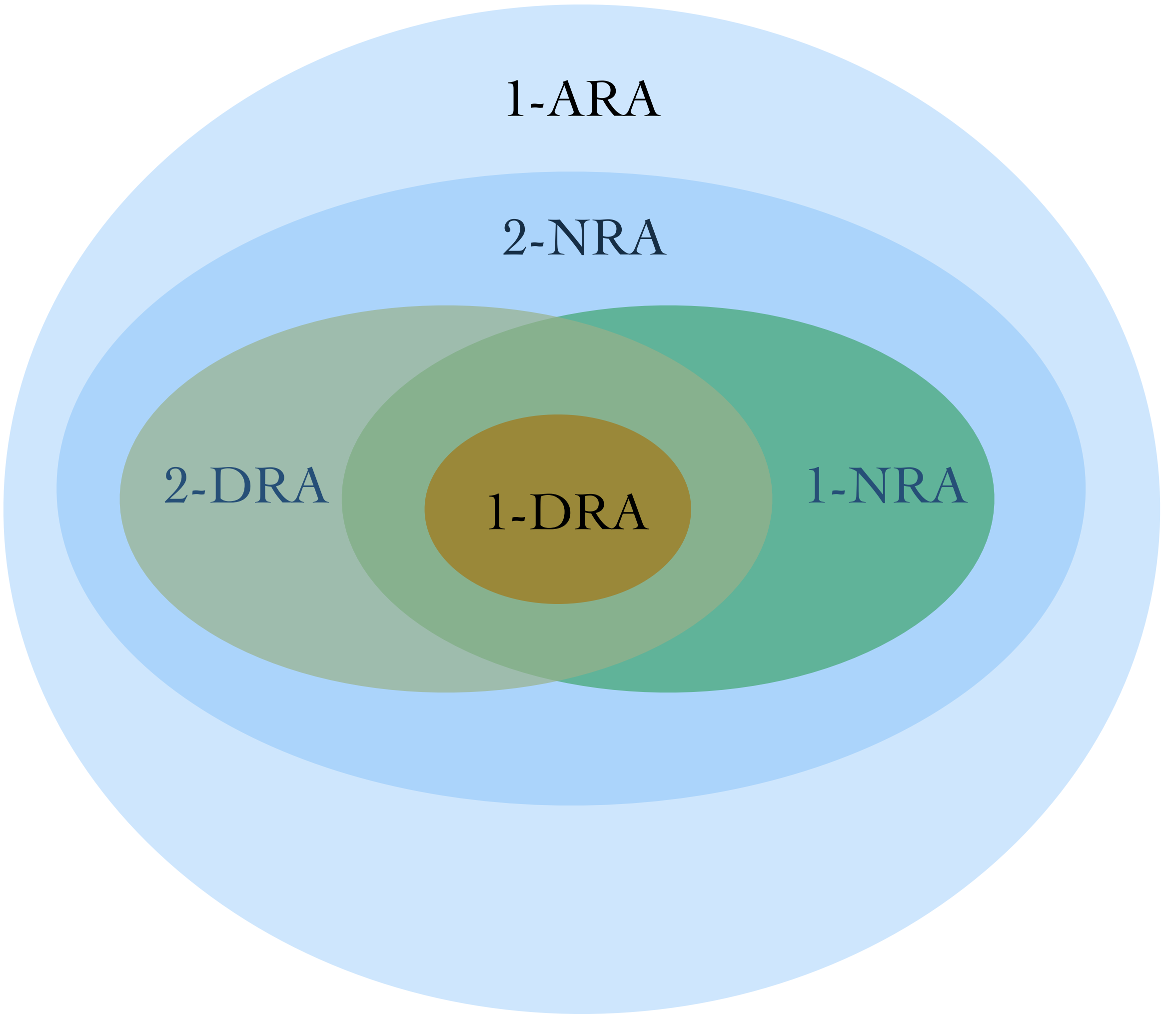
Question (Klin): What should the question be?

Decidability



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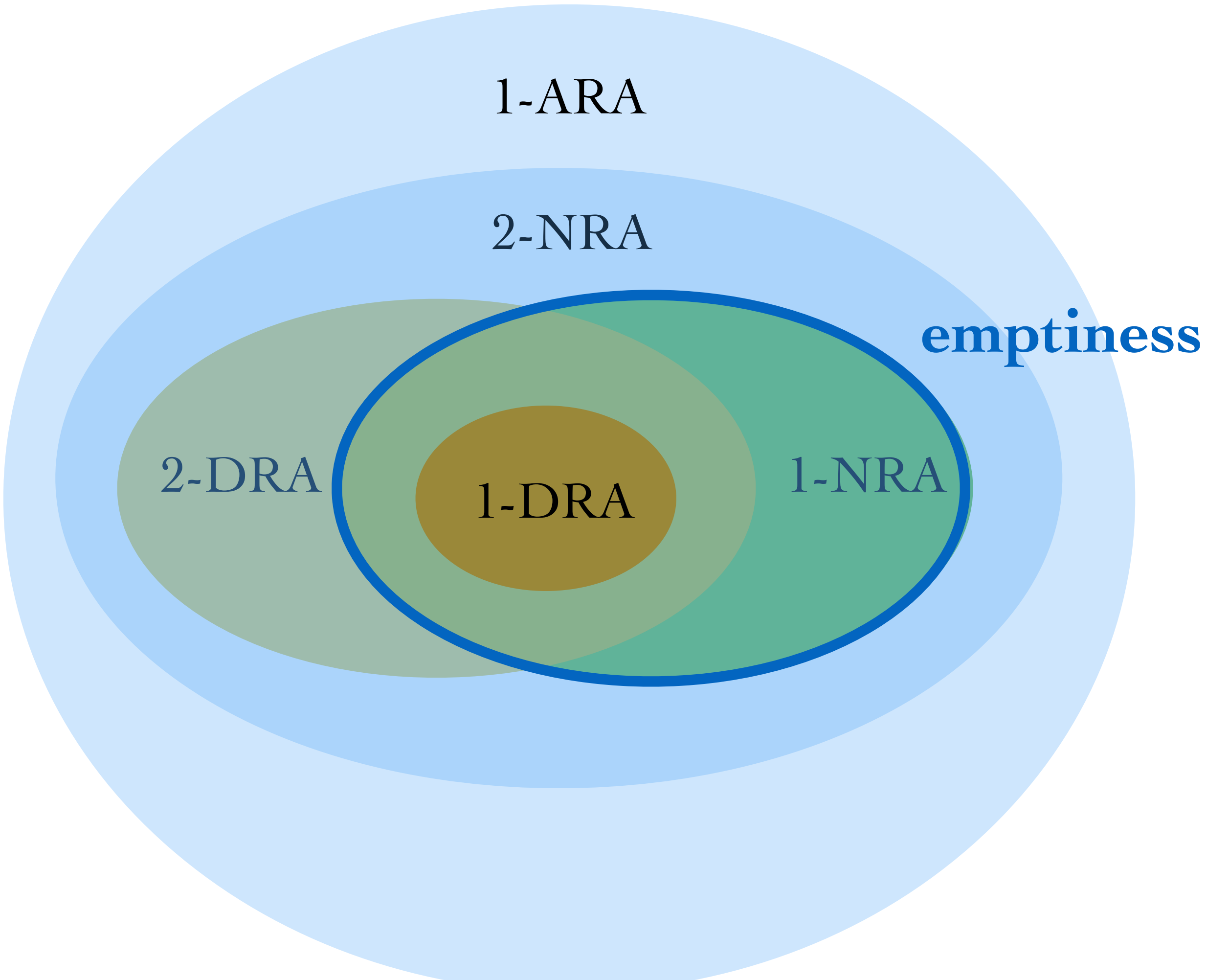

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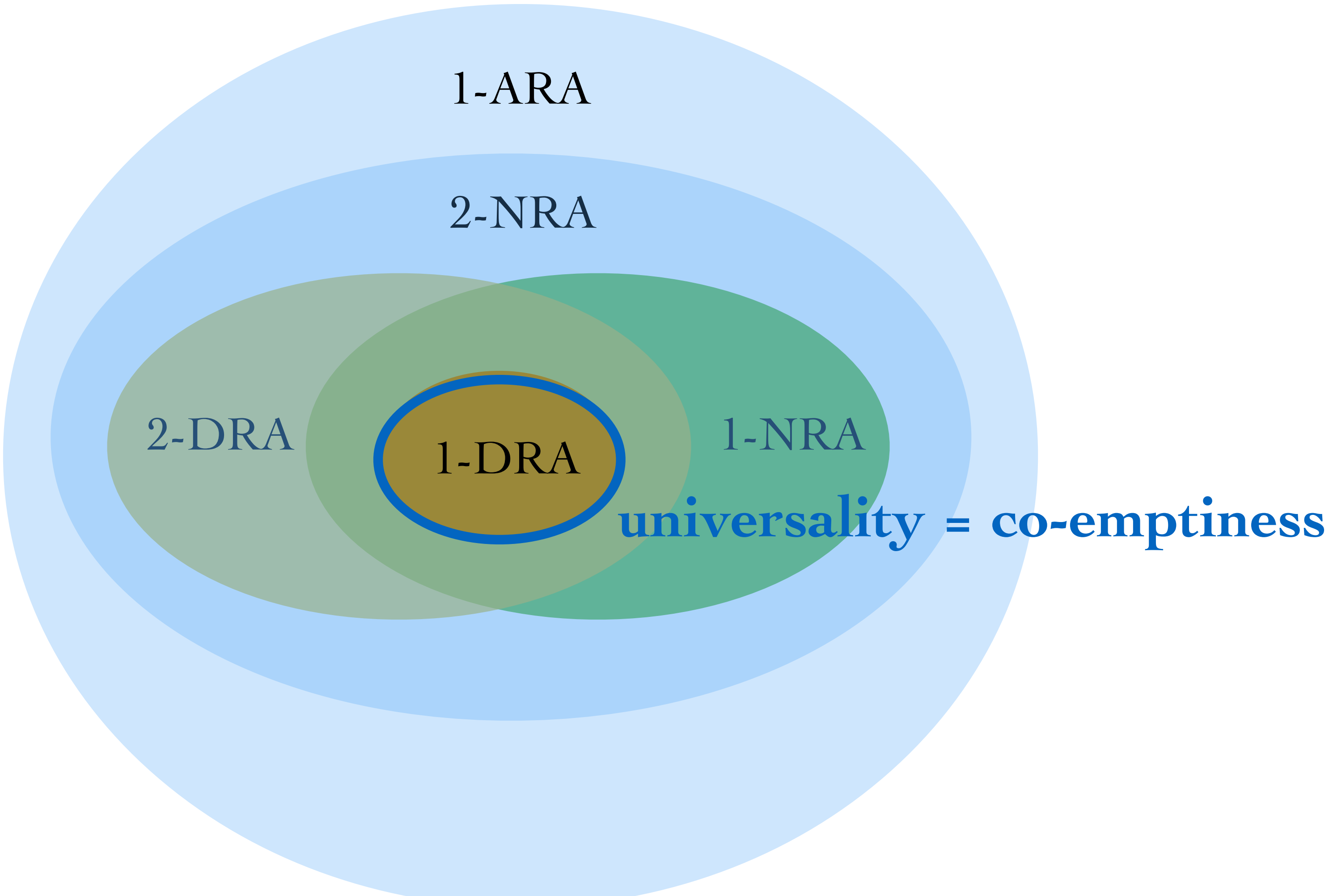

II. Some recent advances

1) Deterministic separability
2) Deterministic collapse
3) Commutative images
4) Single-use registers

1) Deterministic separability

Motivation: DRA are decidable while NRA are not

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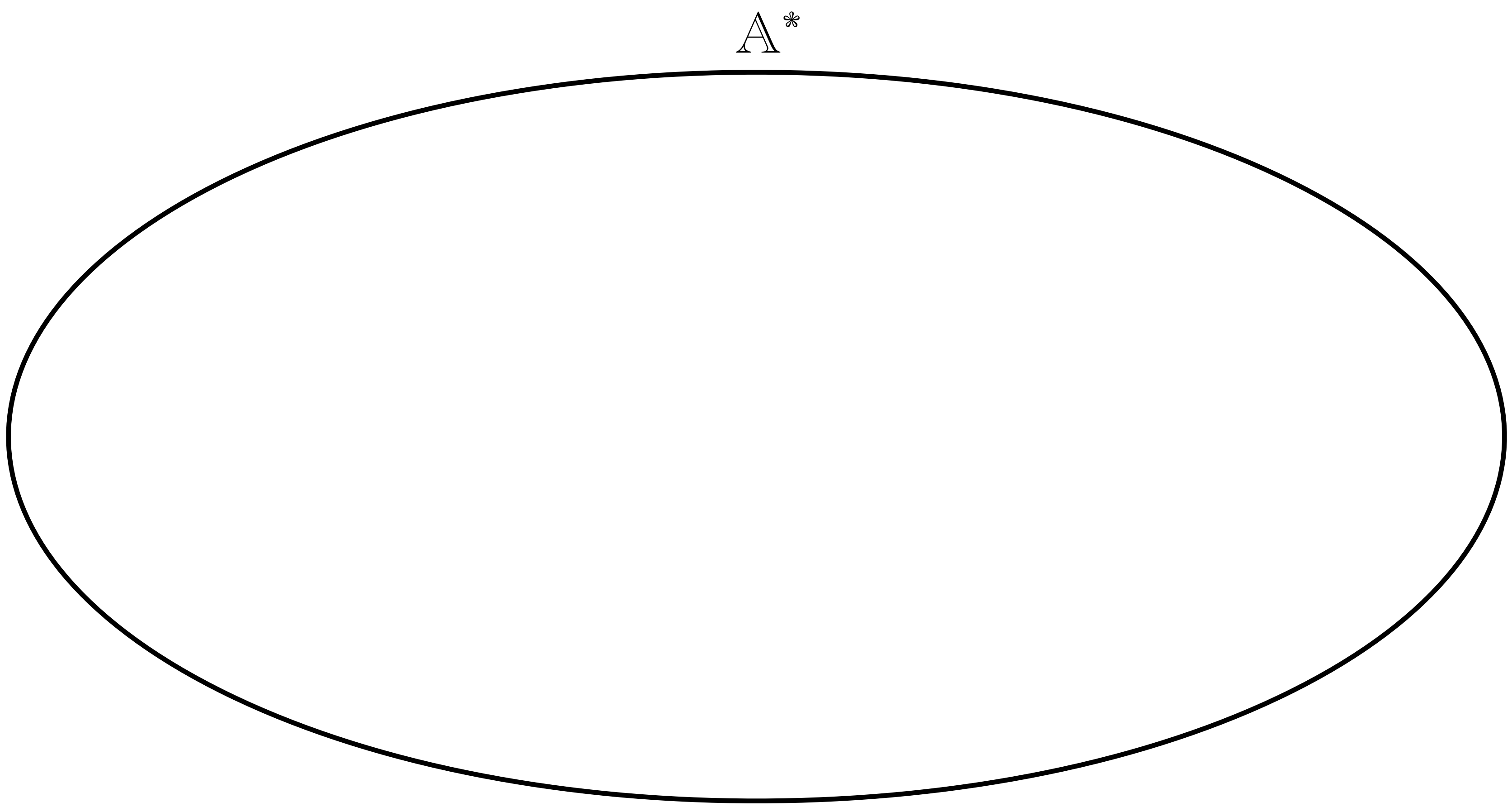

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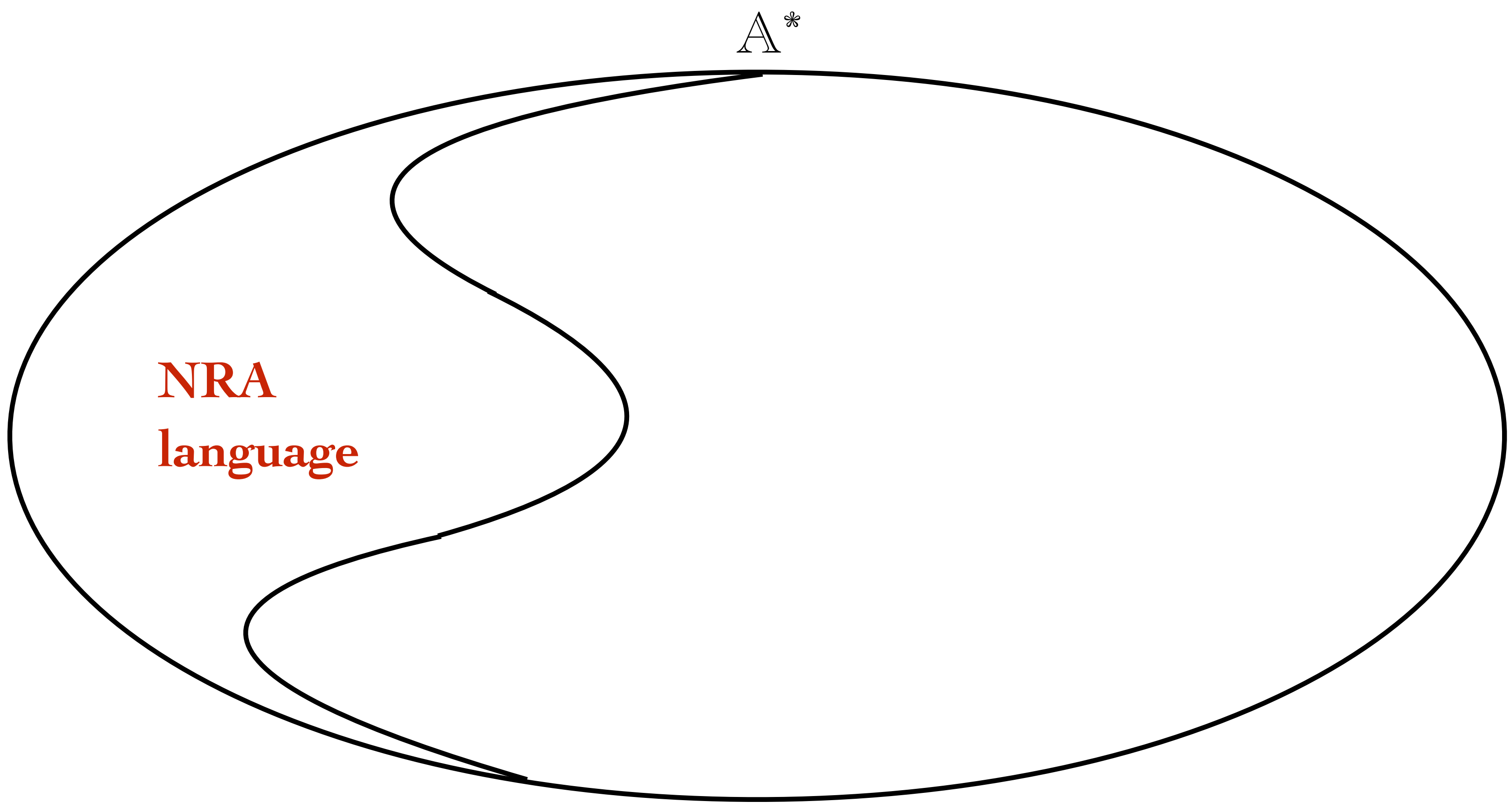

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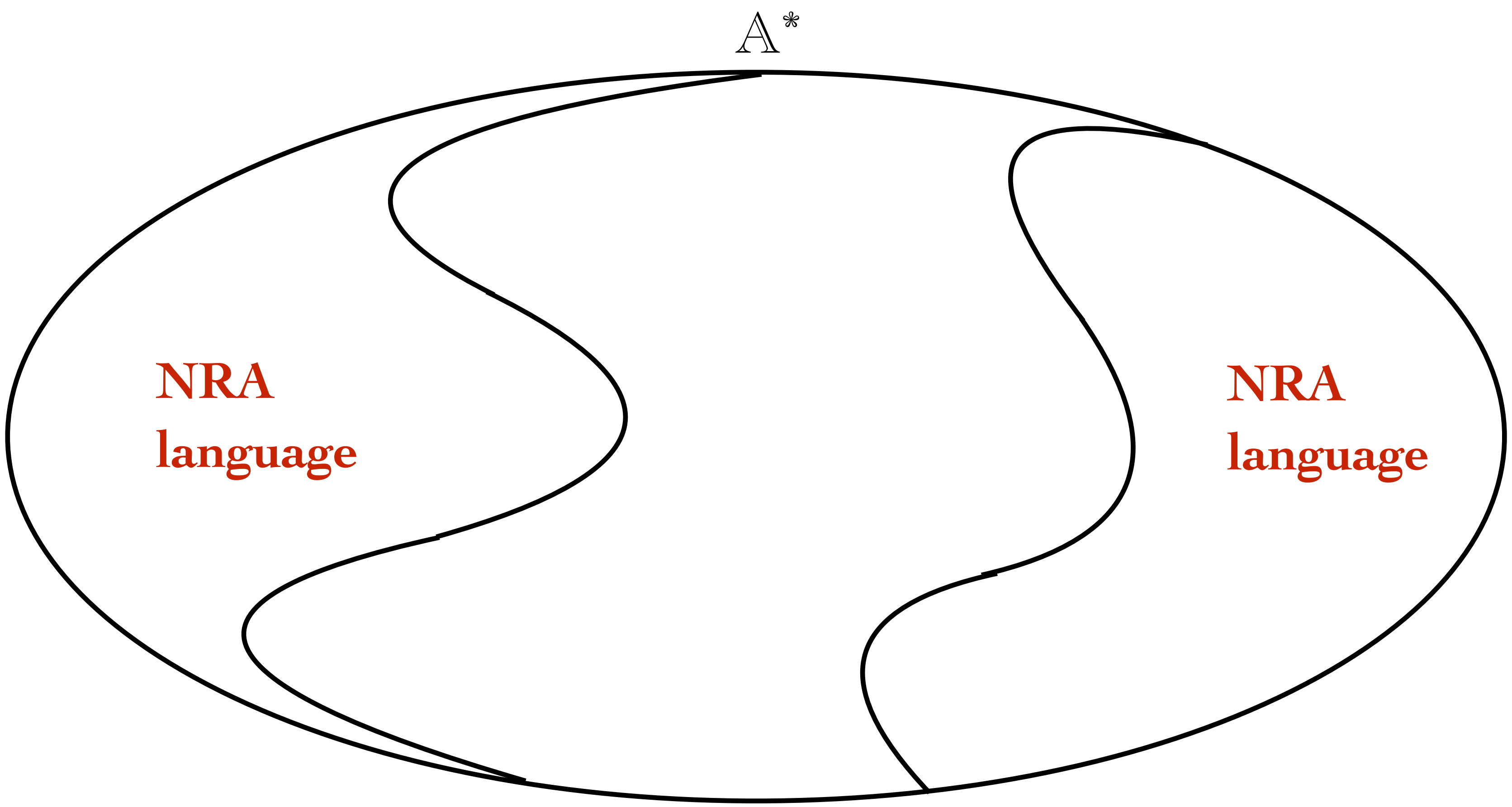

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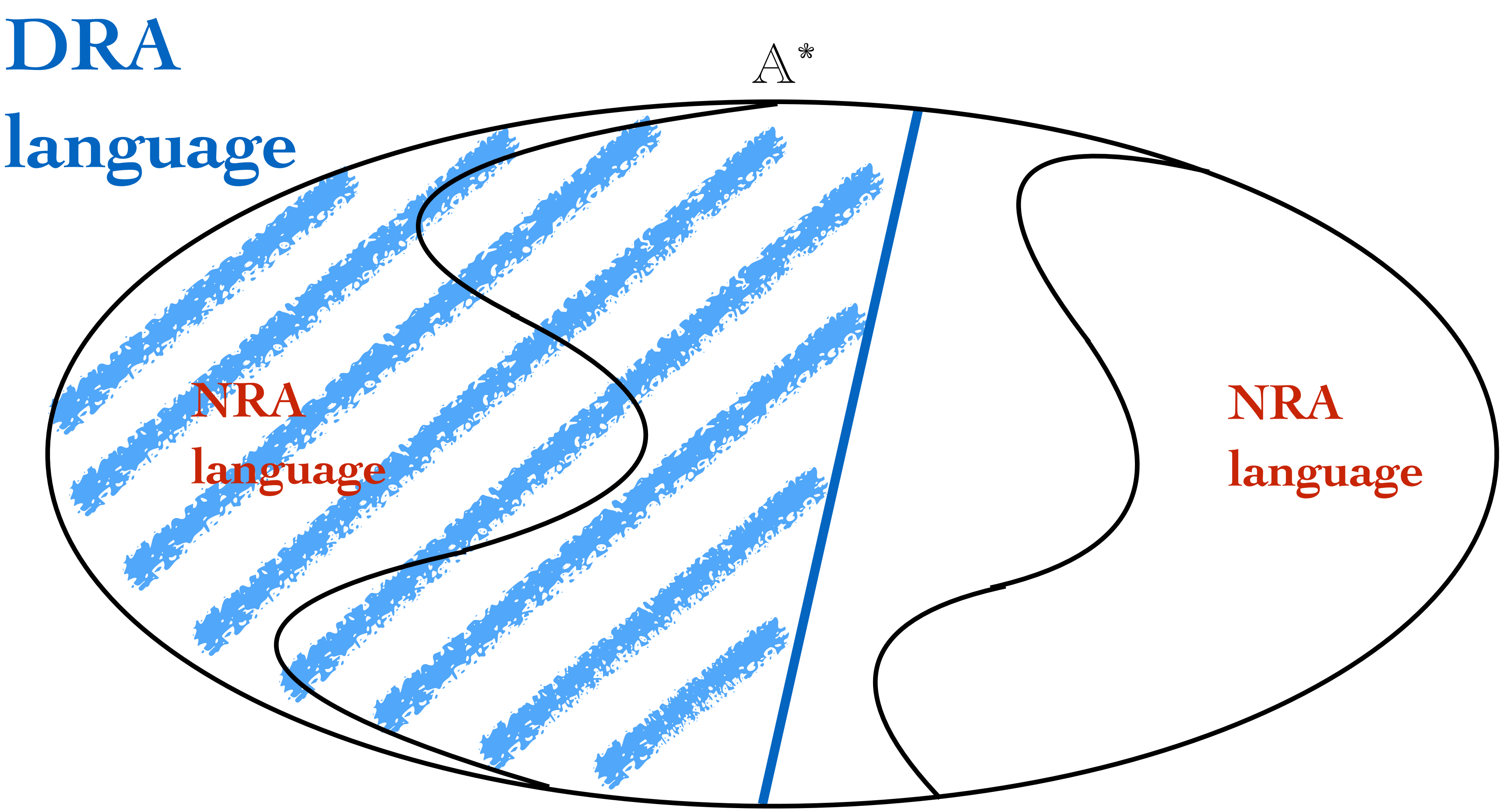

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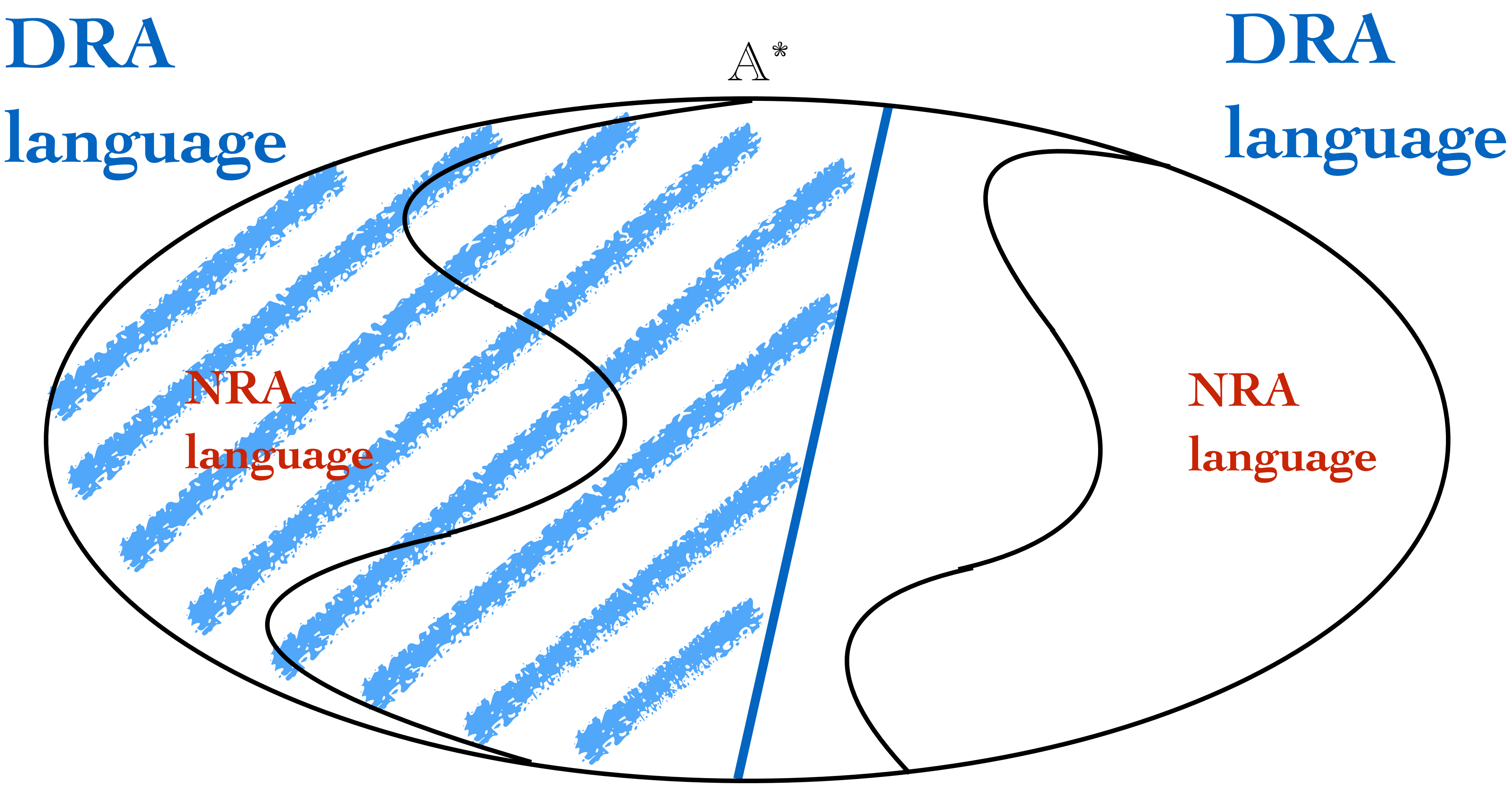

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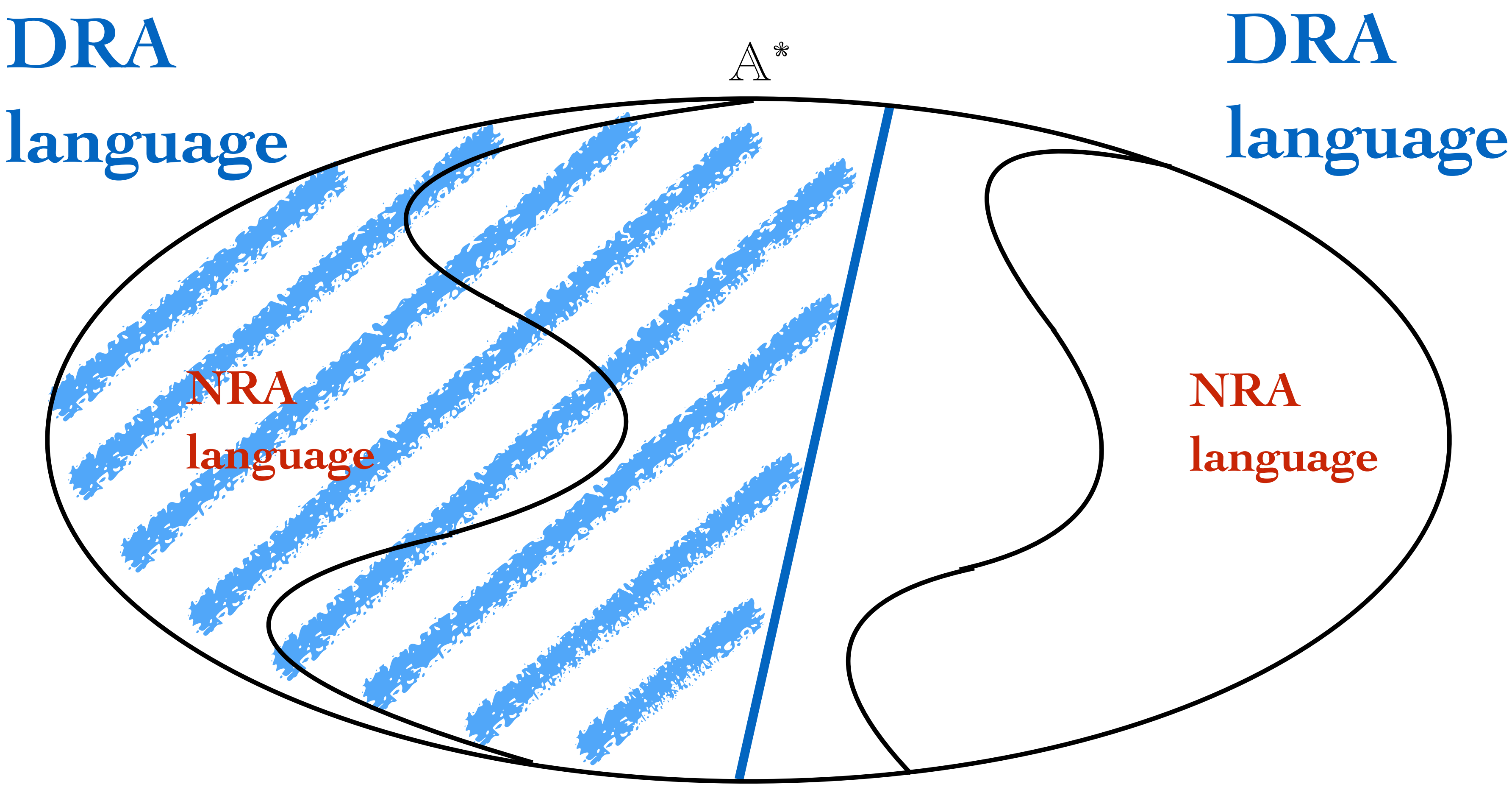
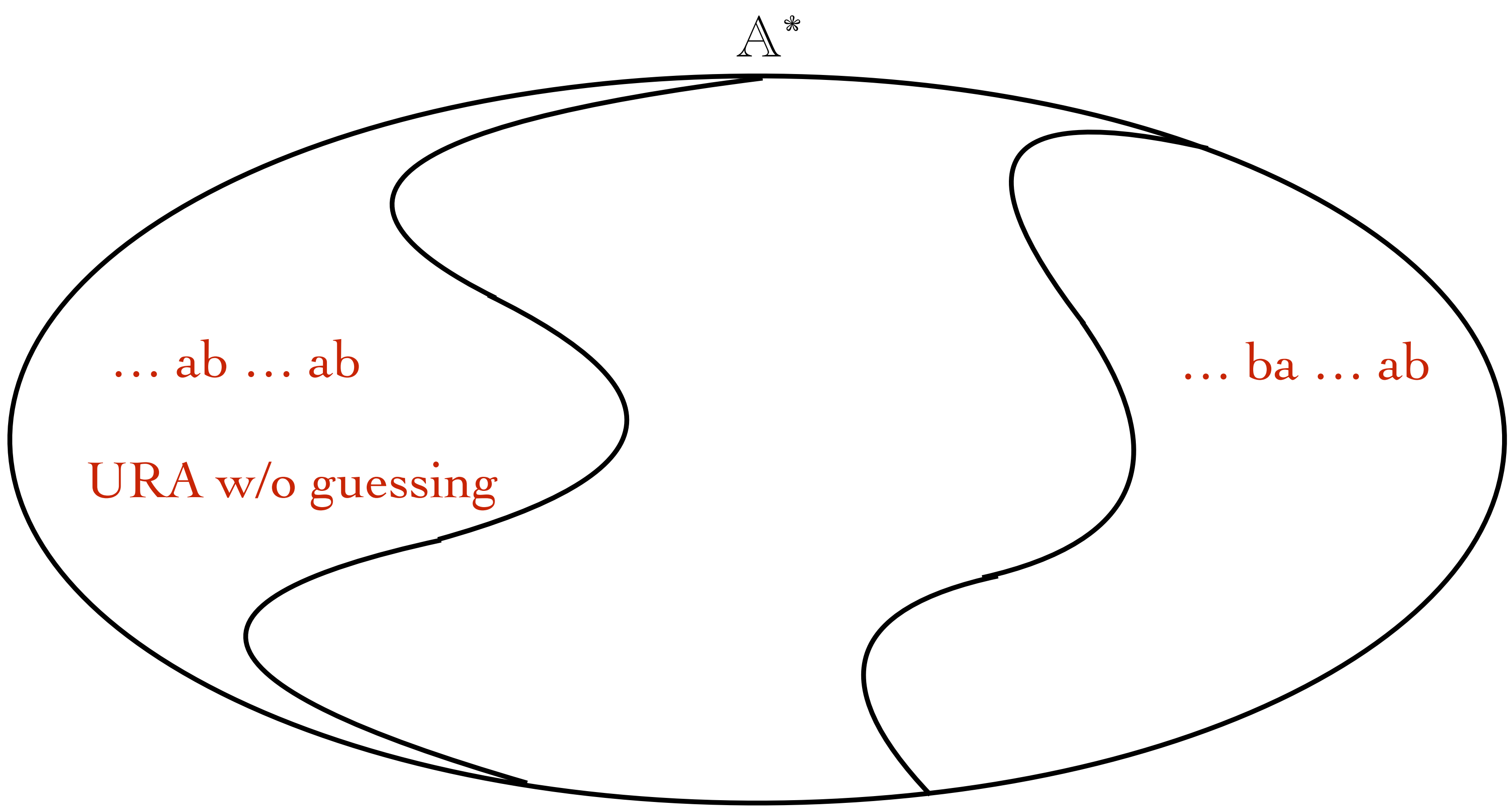


deterministic separator = classifier

1) Deterministic separability

Example: no deterministic separator



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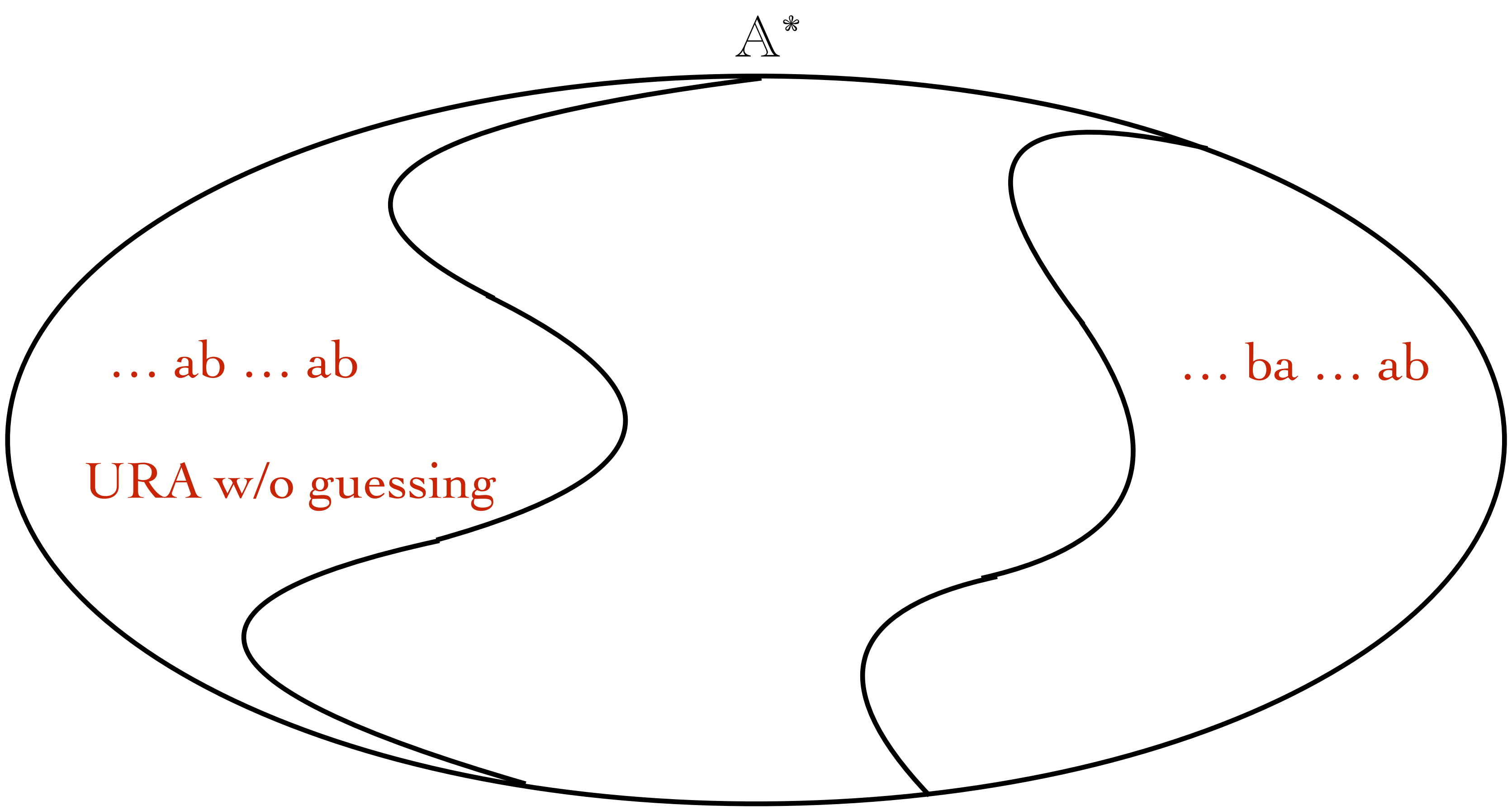
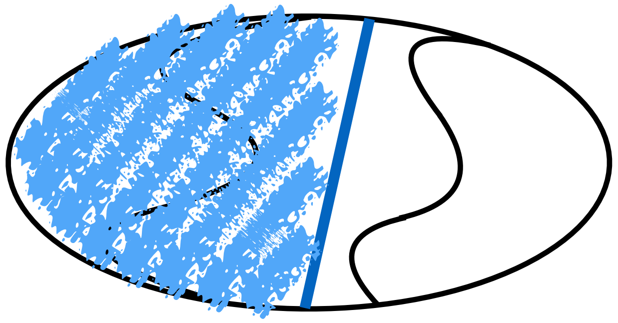


Conjecture (Colcombet 2012): there is always an **unambiguous** separator

1) Deterministic separability



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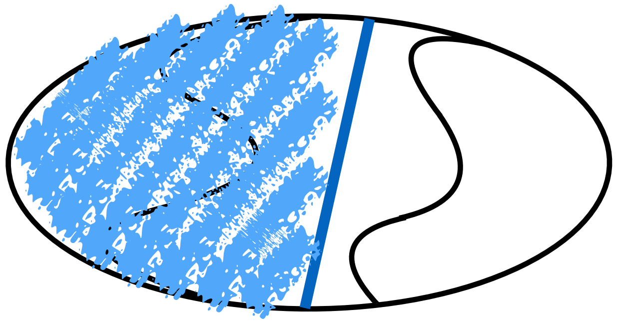

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timed
automata

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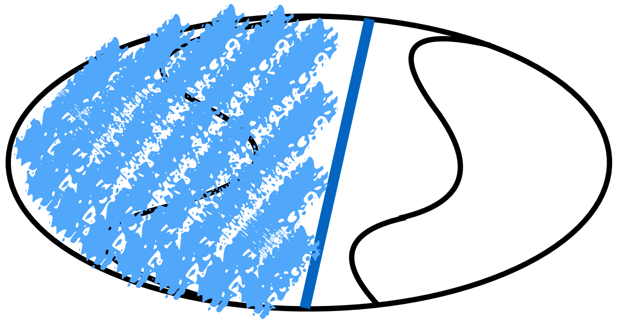

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Alice:

Bob:

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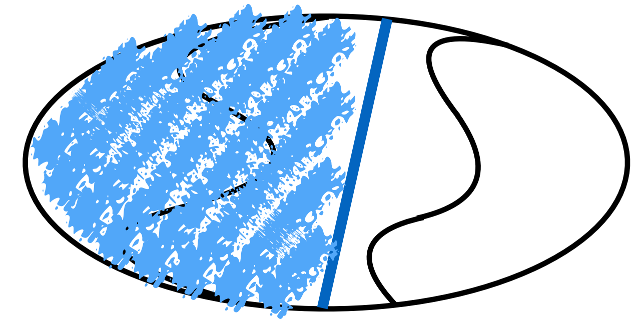

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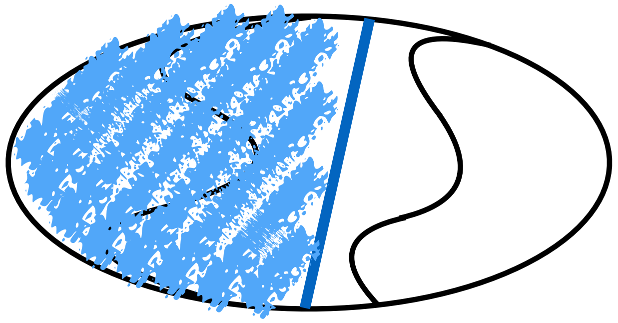

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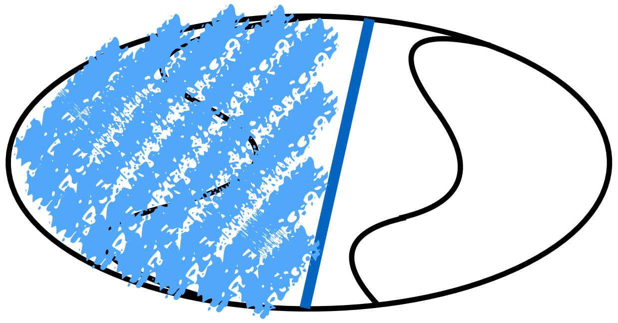

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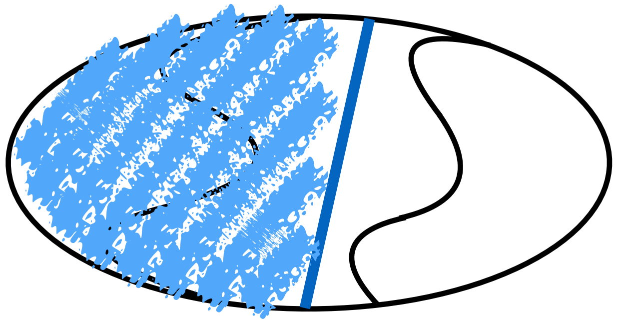

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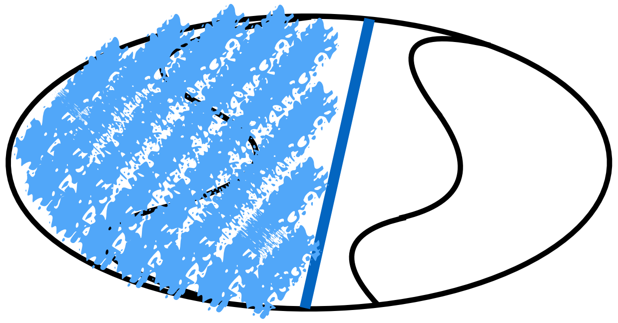

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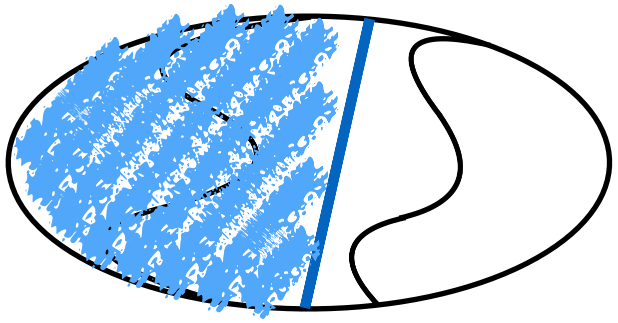

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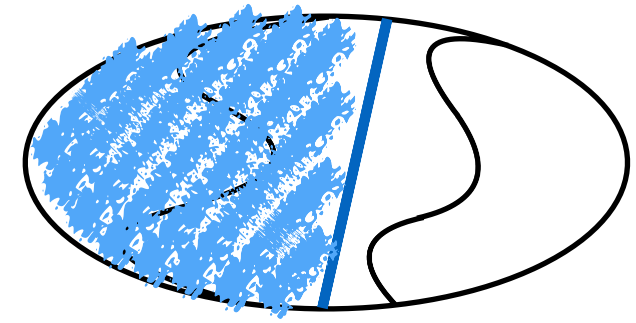

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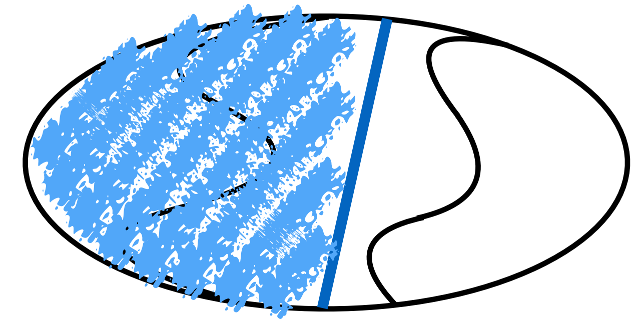

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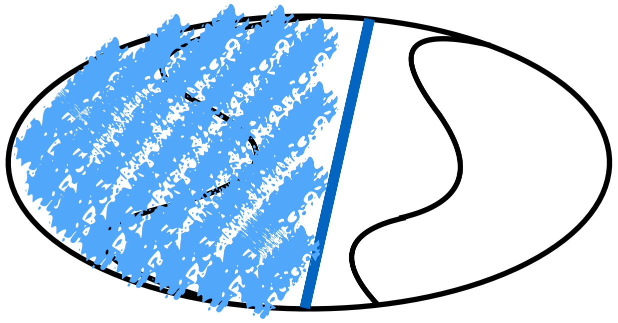

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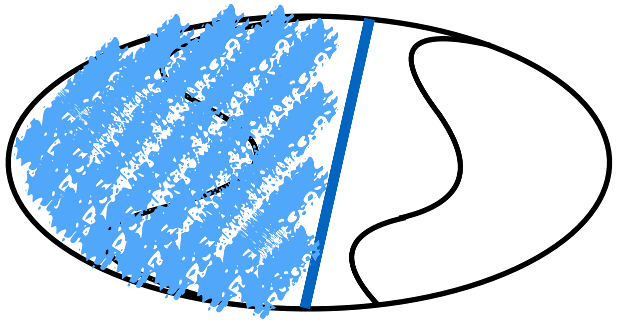

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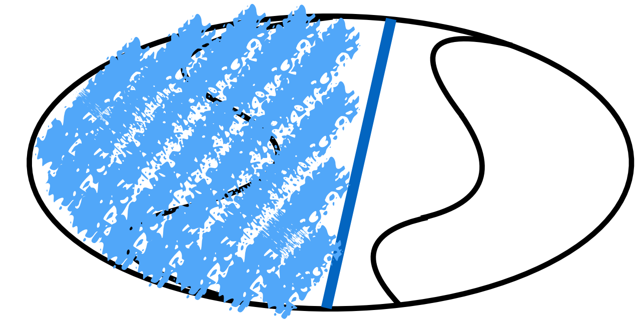

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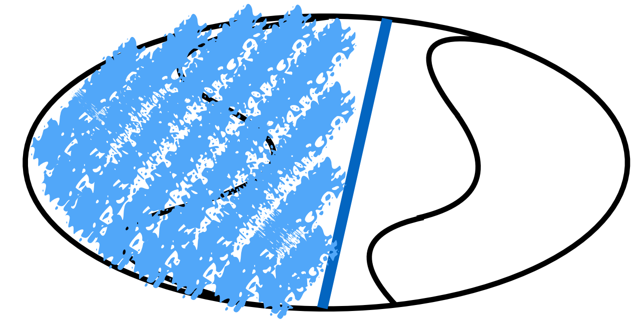

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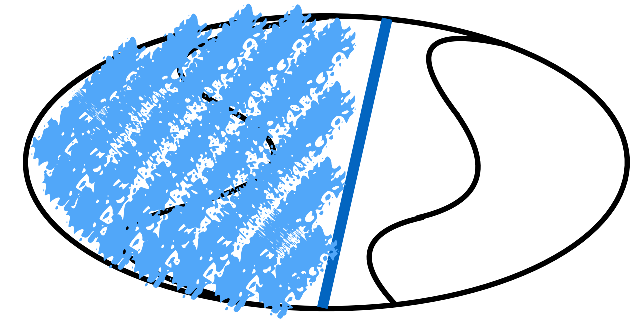

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k-register winning strategy = *k*-DRA separator

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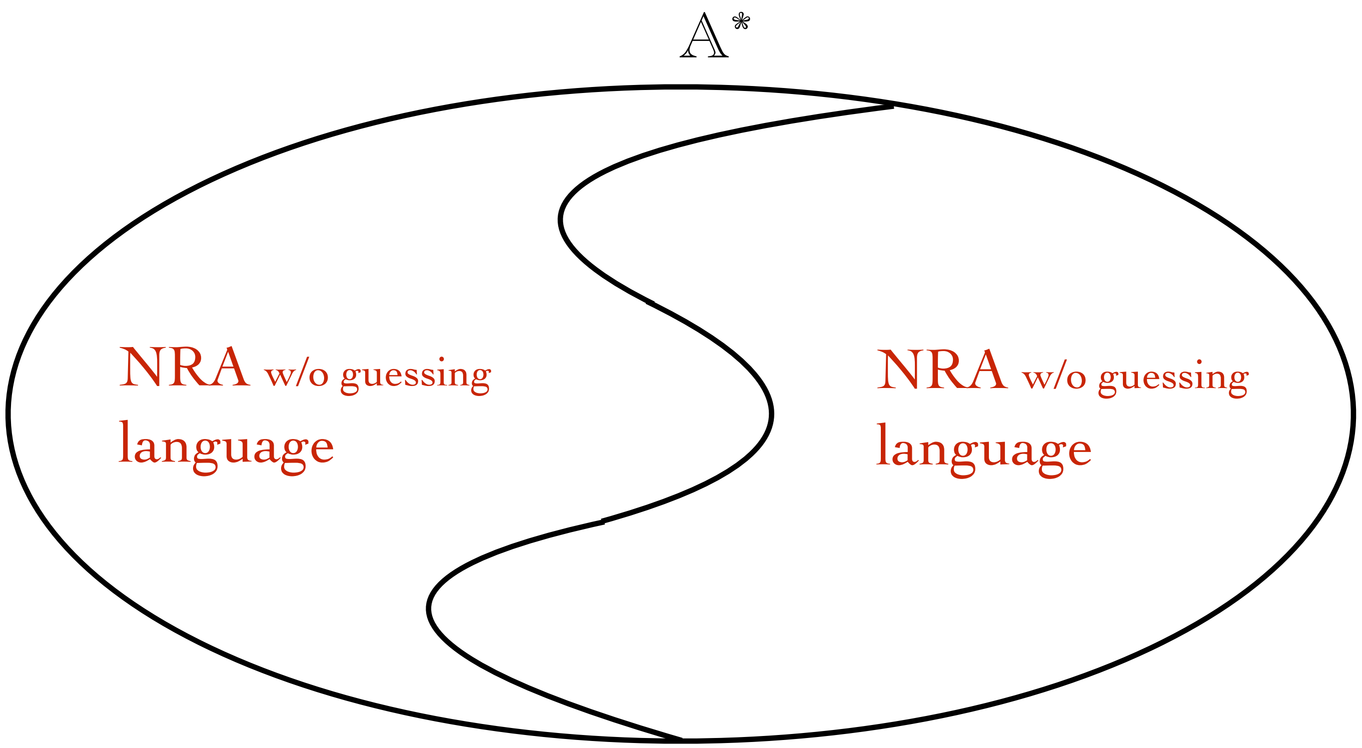
Special case: two complementing DRA



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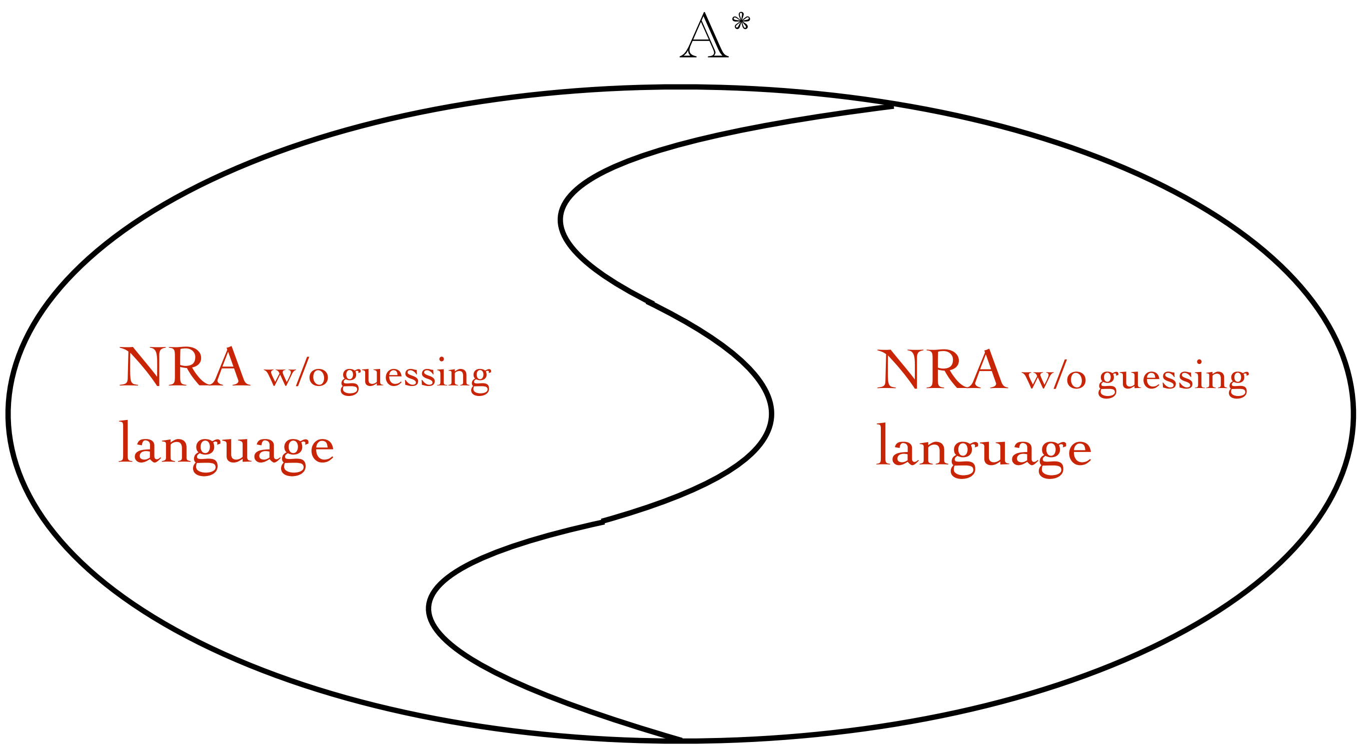

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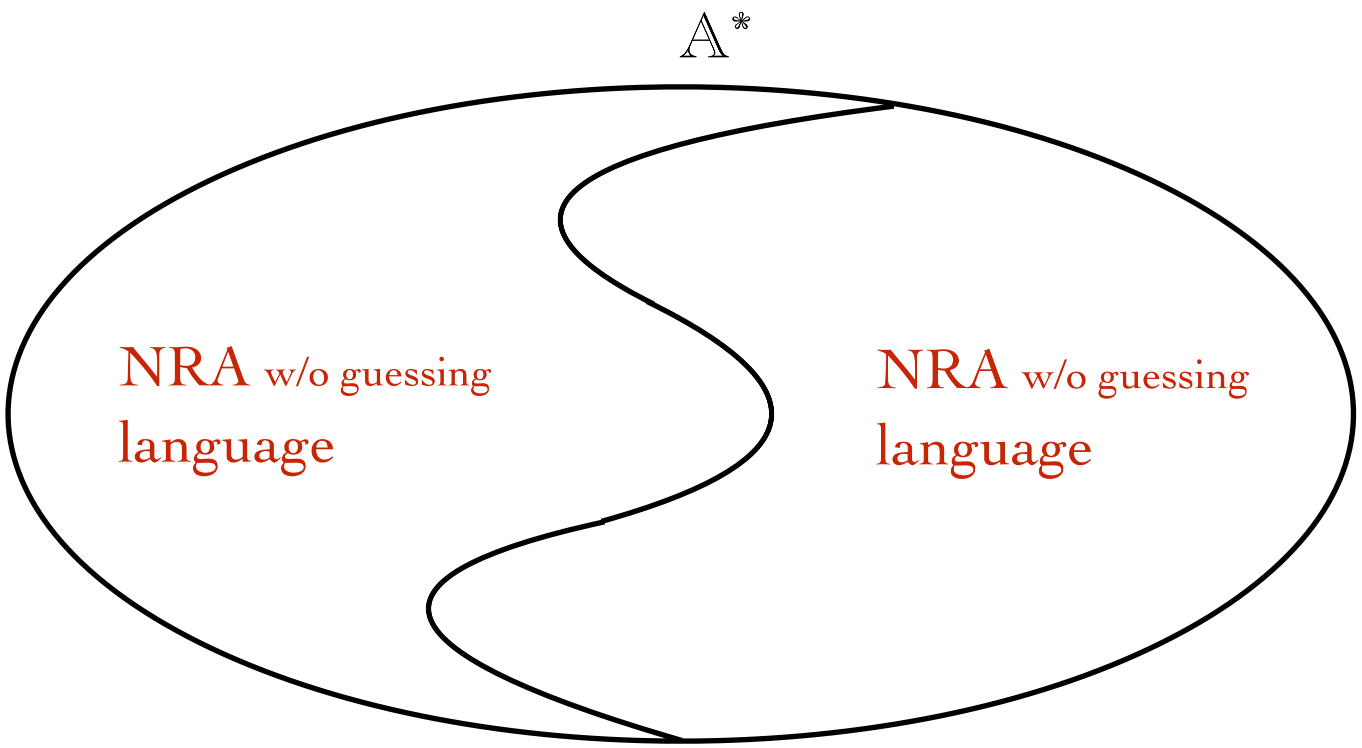

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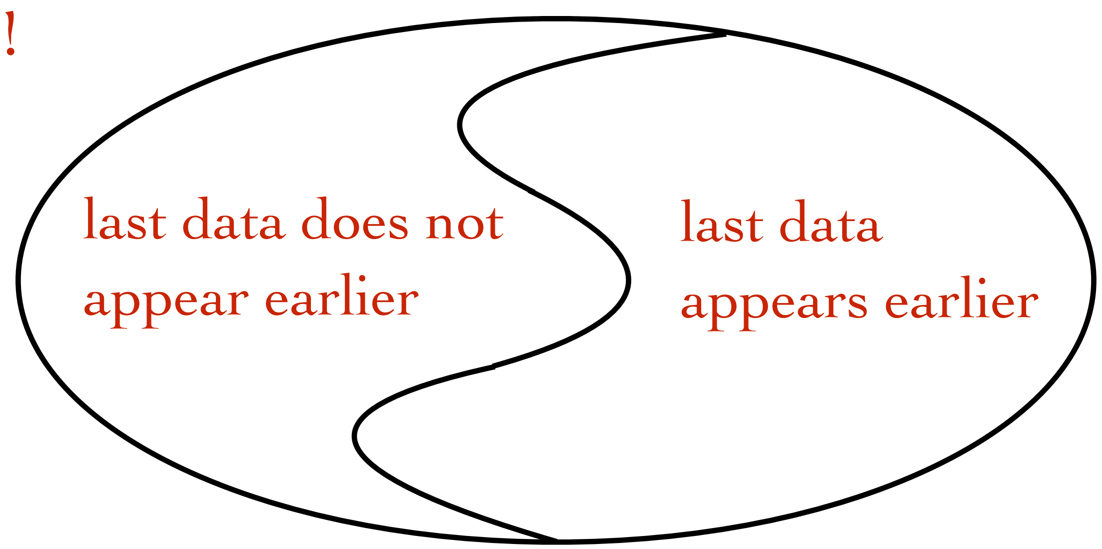
Fails with guessing!



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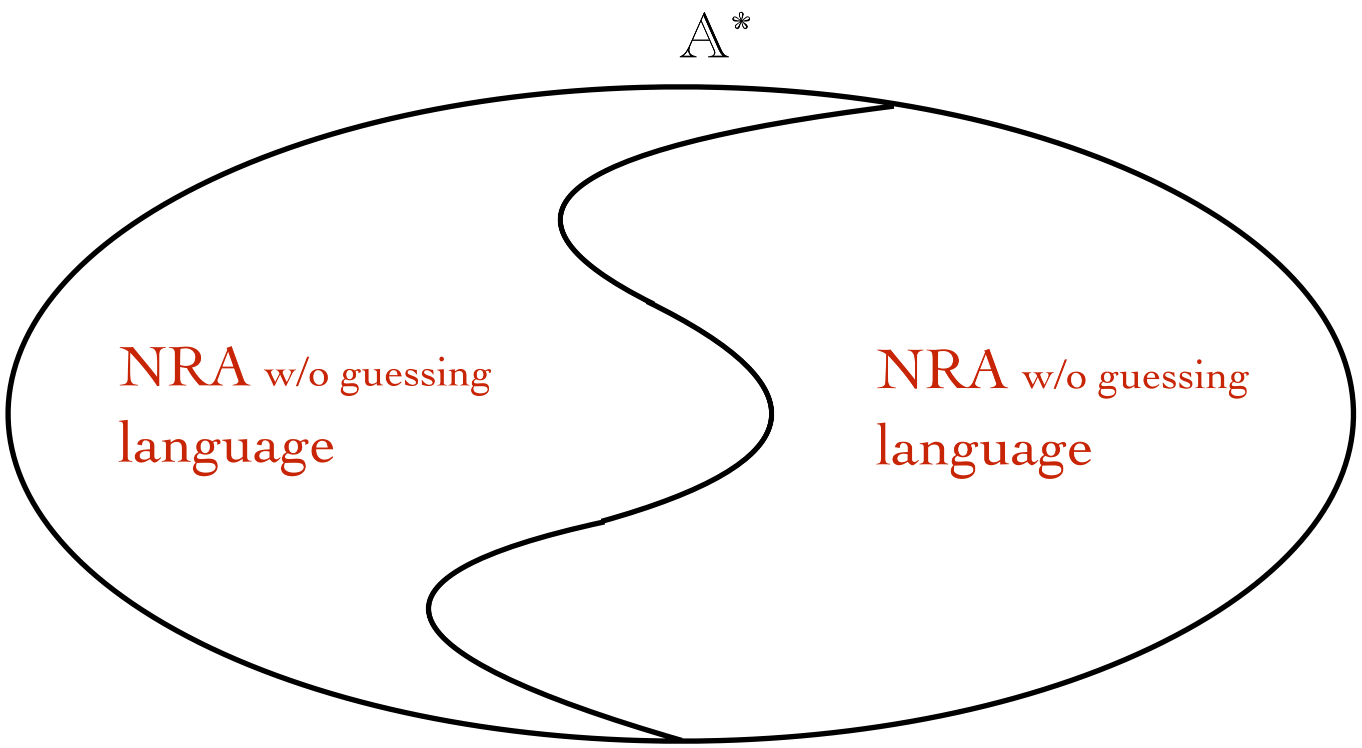

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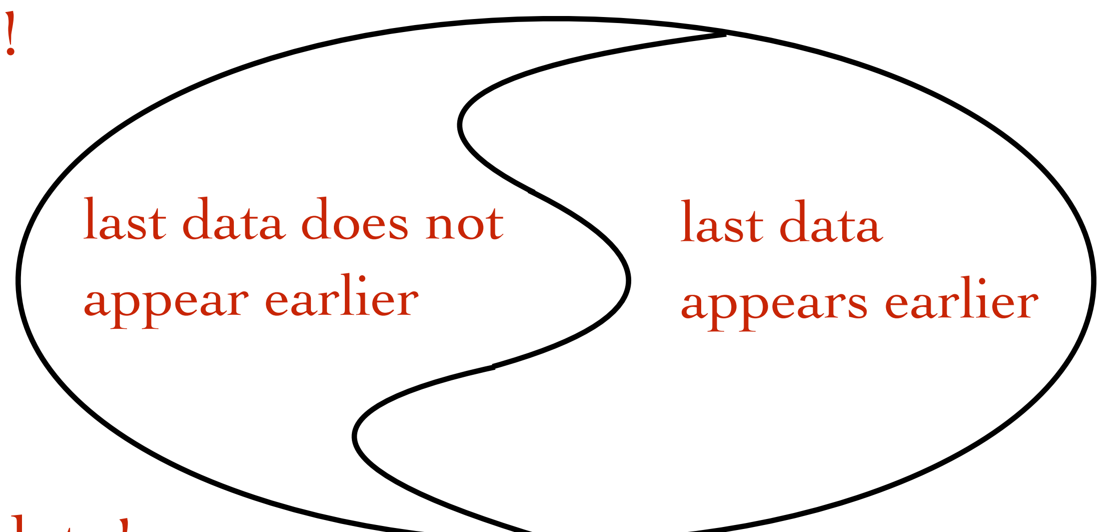

Fails with ordered data!

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Key tool: splitting lemma (+ orbit-finite Myhill-Nerode)

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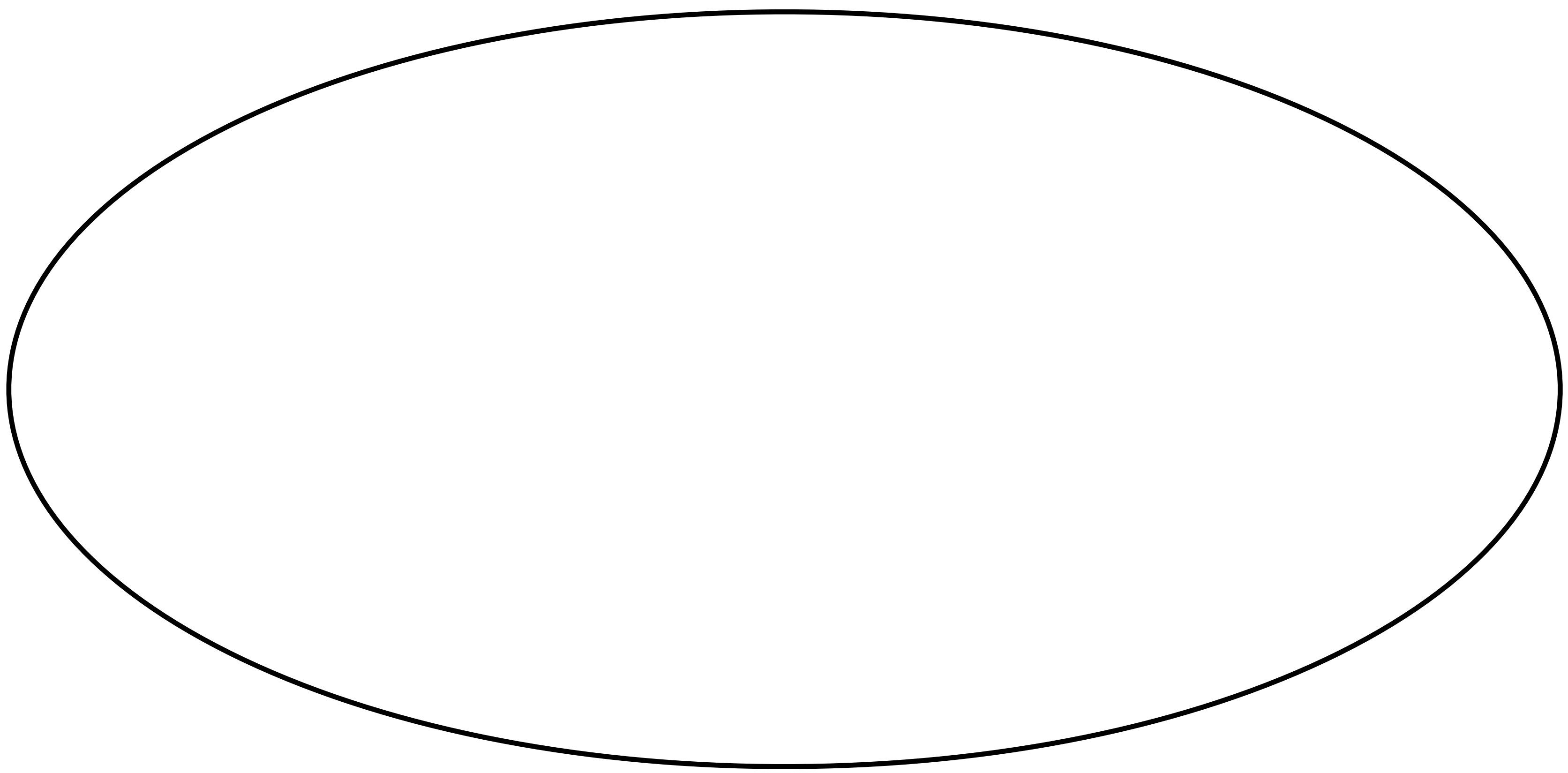
an invariant set



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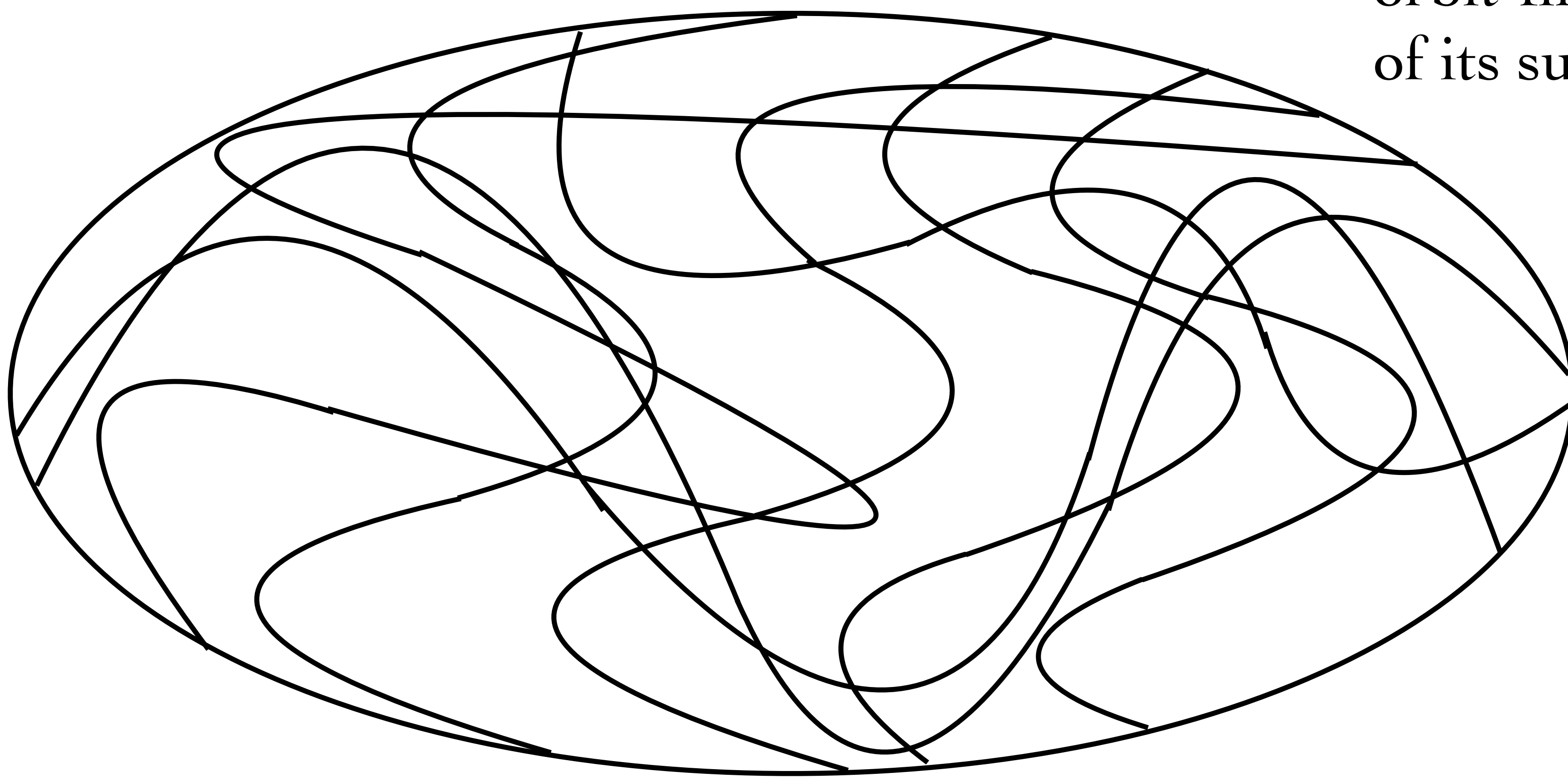

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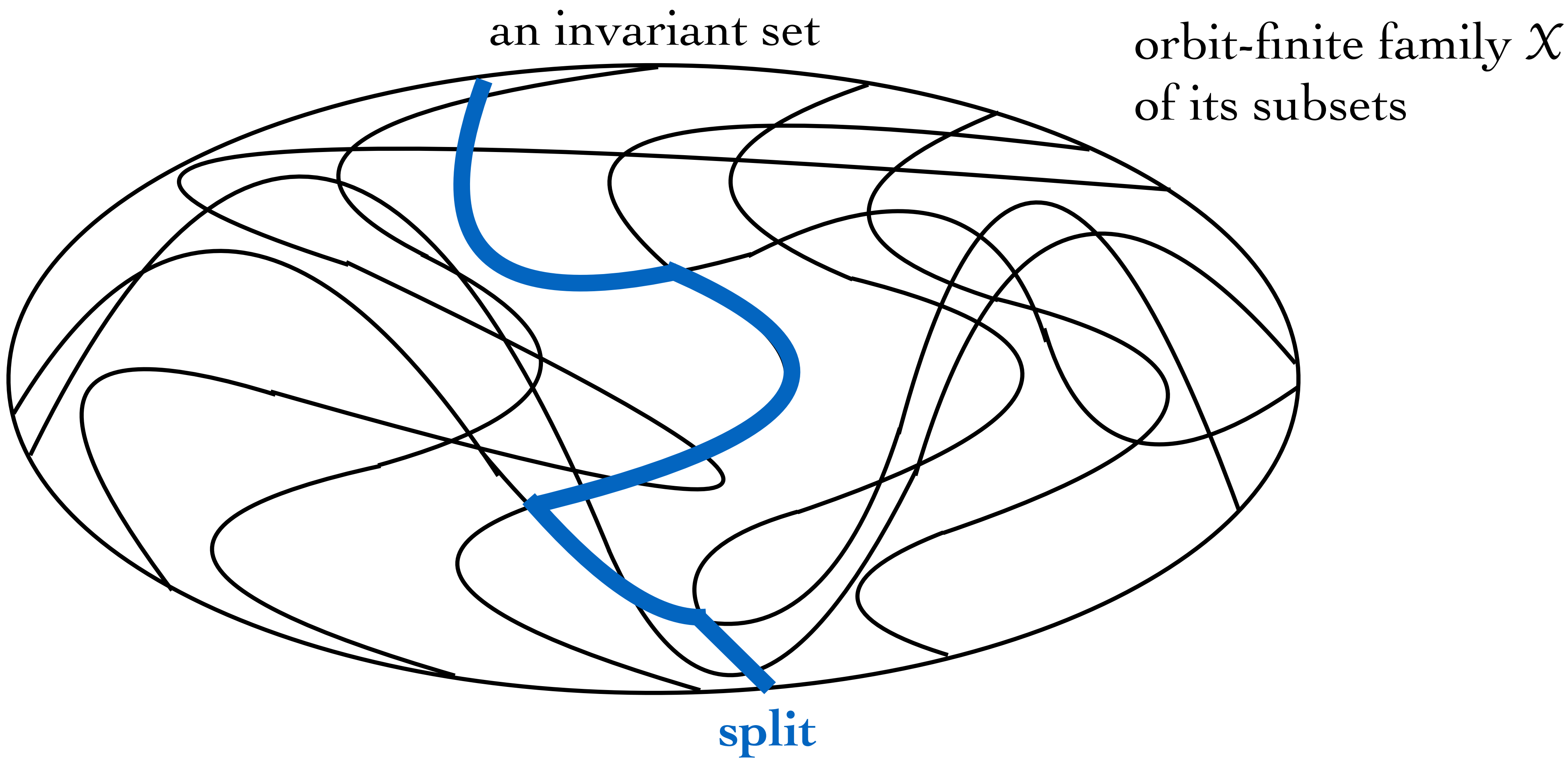

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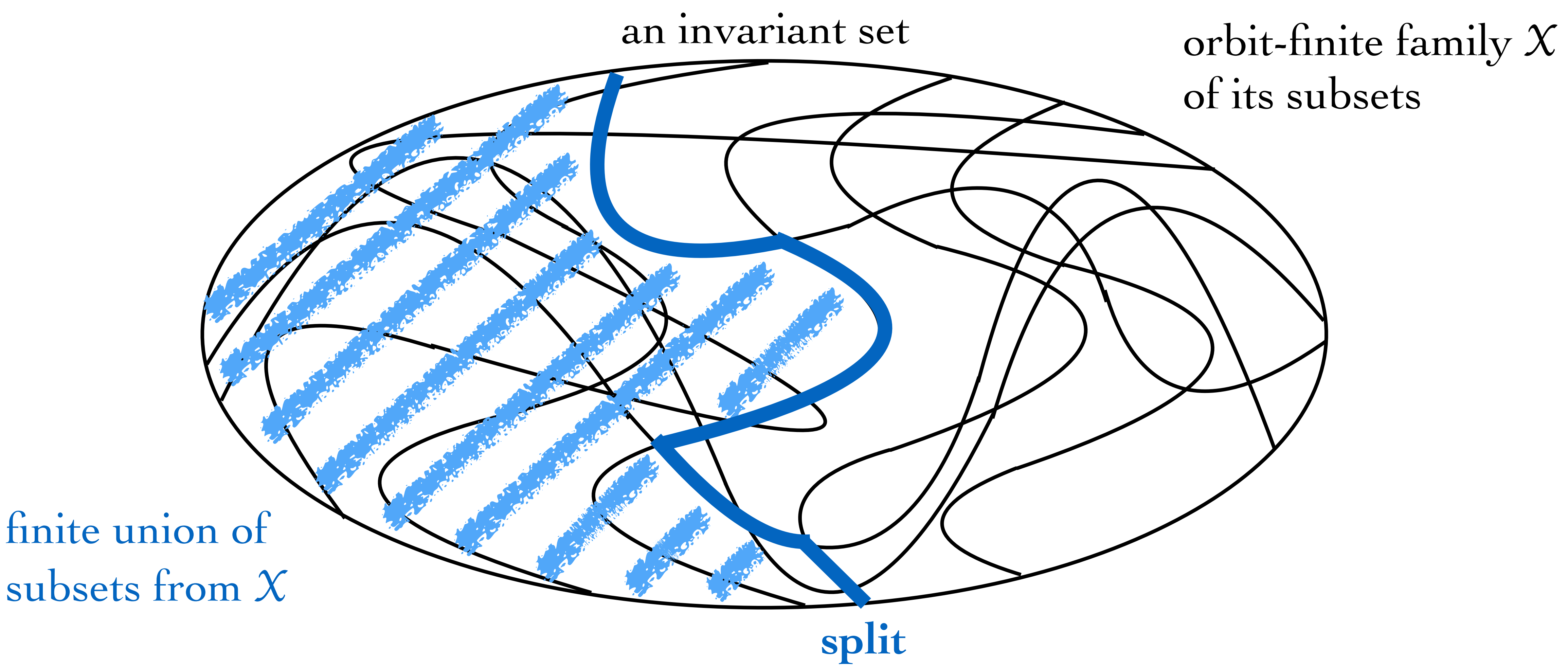

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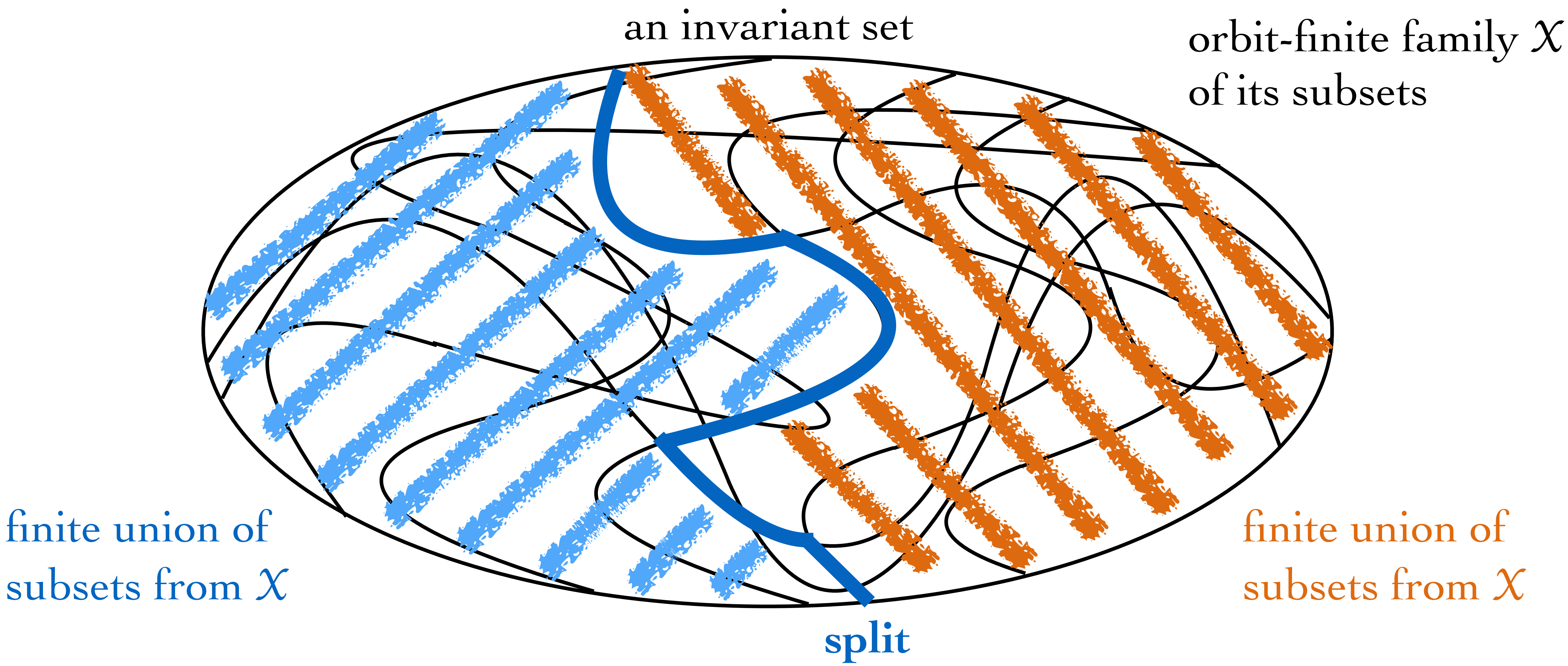

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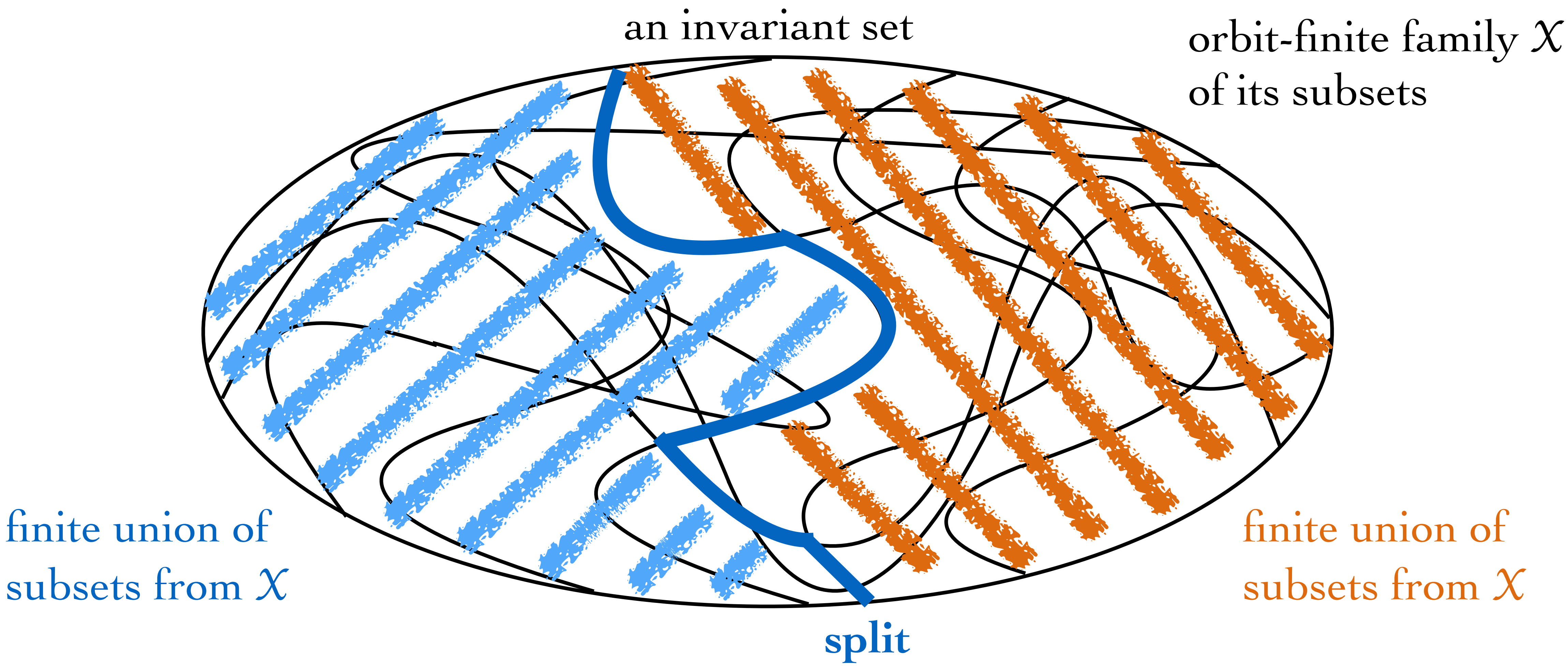

Splitting Lemma: each orbit-finite family $\mathcal{X}$ admits only orbit-finitely splits

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Orbit-finite rational (regular) expressions:

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and even semi-linear

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Theorem (Hofman, Juzepczuk, L., Pattathurajan 2021):
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Key tool: sufficient condition for Hamiltonian cycle
in a strongly connected directed graphs

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Bojańczyk, Stefański 2020

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a		
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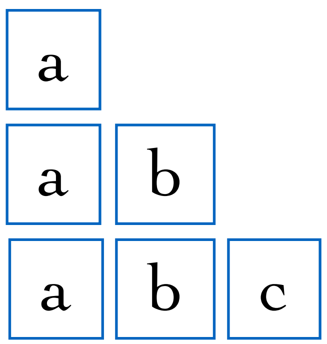

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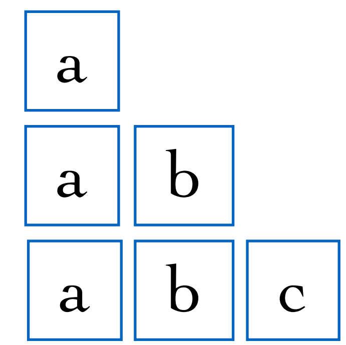

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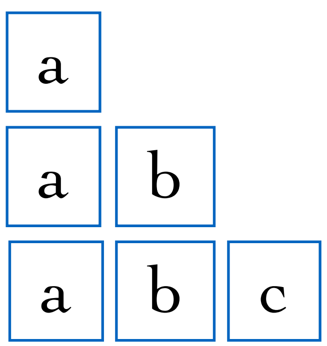

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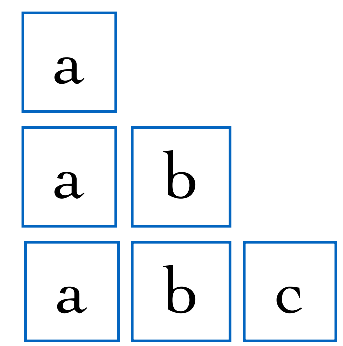

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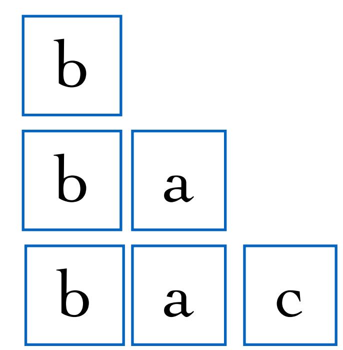

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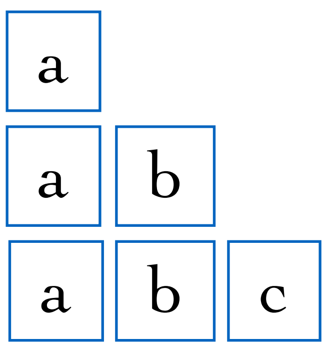

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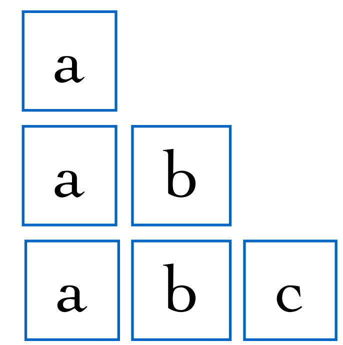

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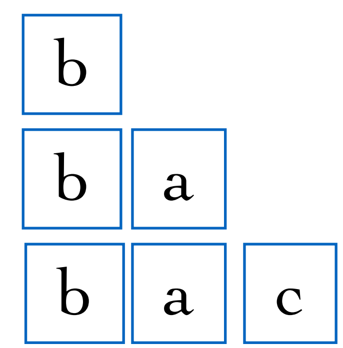

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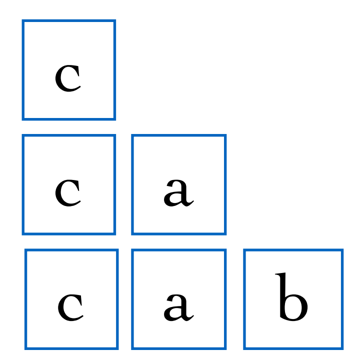

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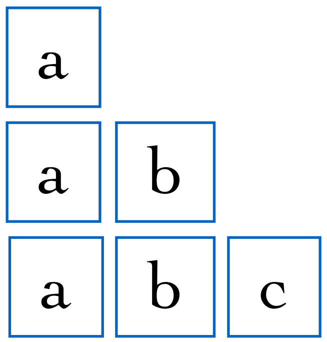

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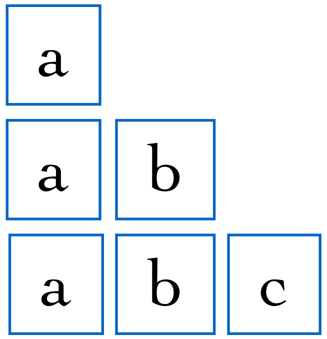

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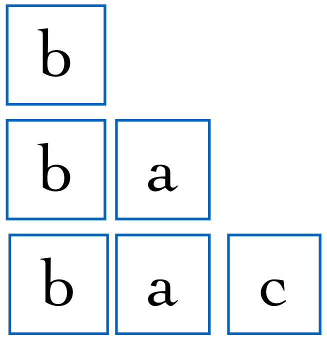

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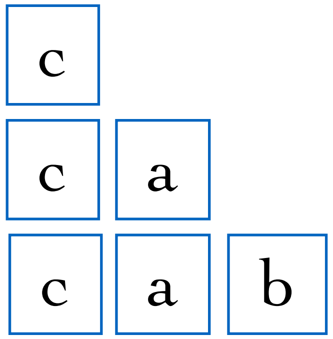

- otherwise reject

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Starting point:

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Key tool: generalise the standard transformation of 2-way to 1-way automata, to register automata

I. Introduction to register automata

II. Some recent advances

1) Deterministic separability
2) Deterministic collapse
3) Commutative images
4) Single-use registers

thank you!

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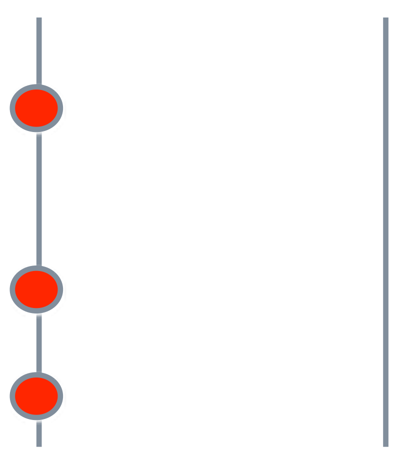

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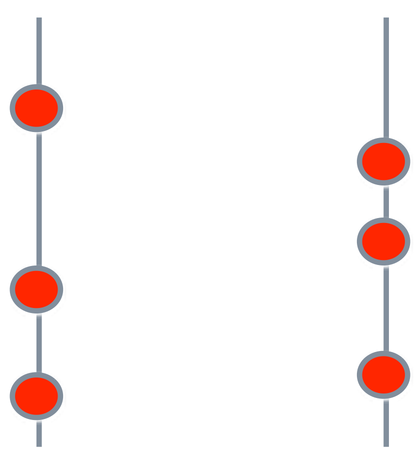

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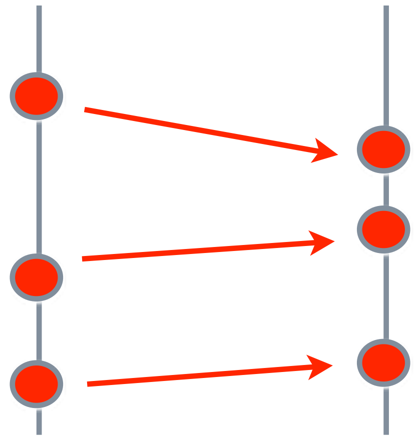

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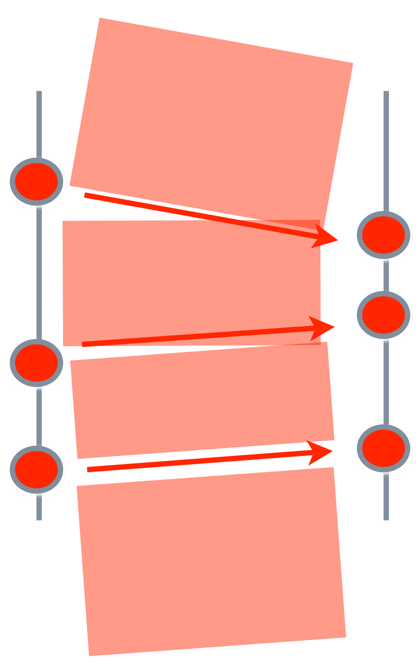

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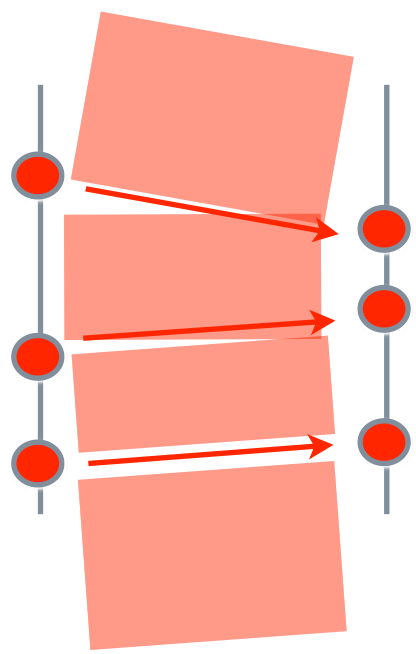

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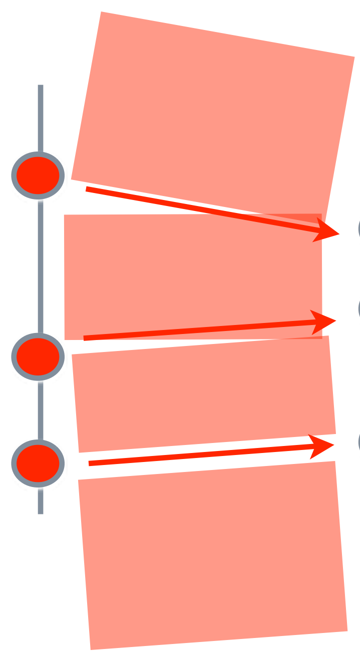

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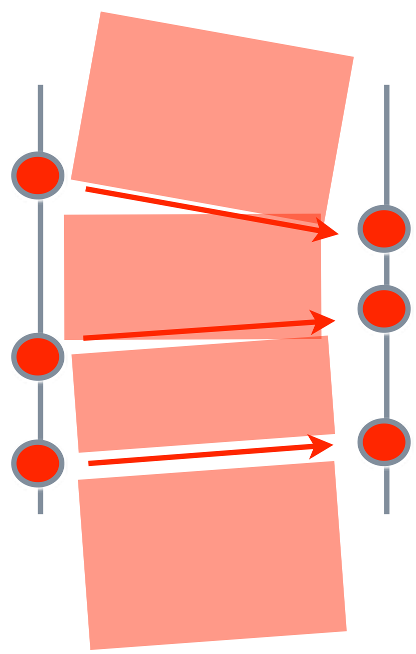

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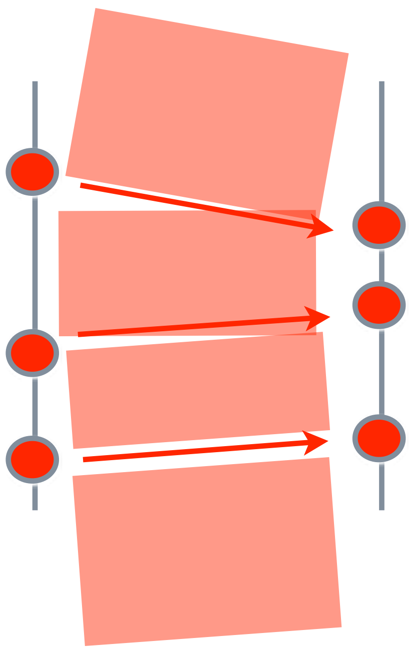

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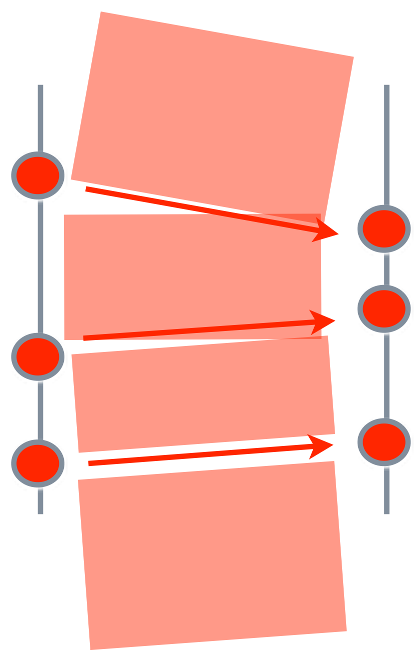

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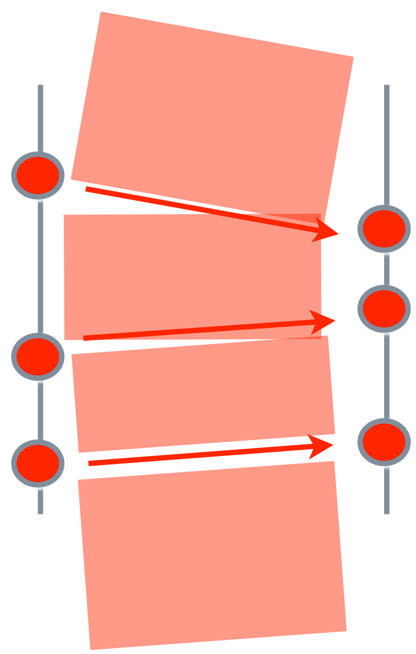

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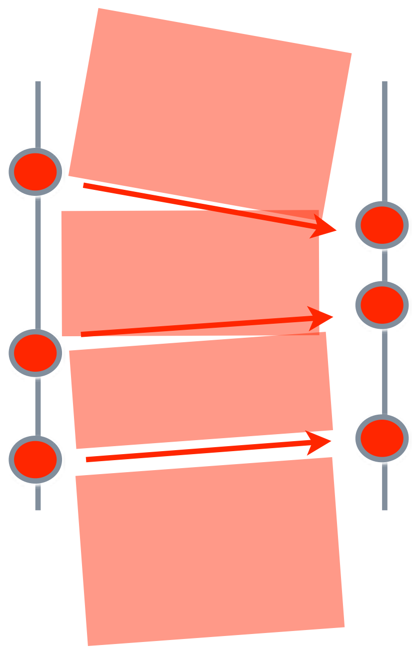

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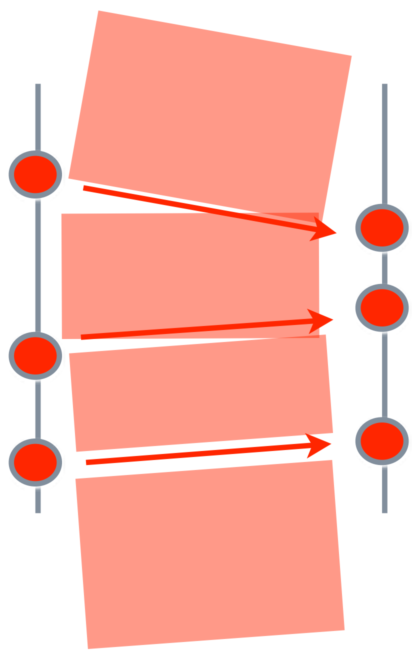

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equality atoms ($\mathbb{N}$, =)
random graph
...

random graph = countable infinite graph yielded almost surely if every pair of nodes is connected by an edge with independent probability ⅓

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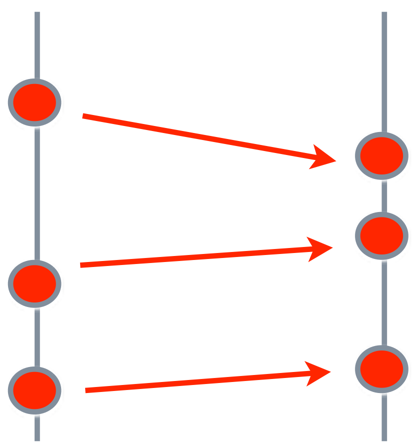

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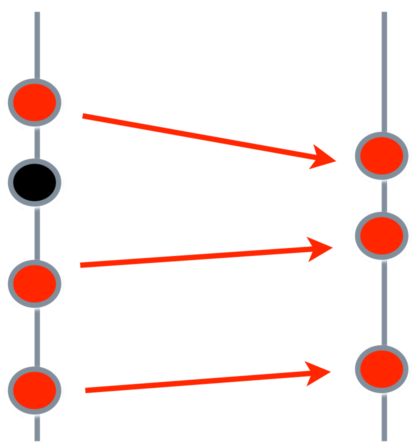

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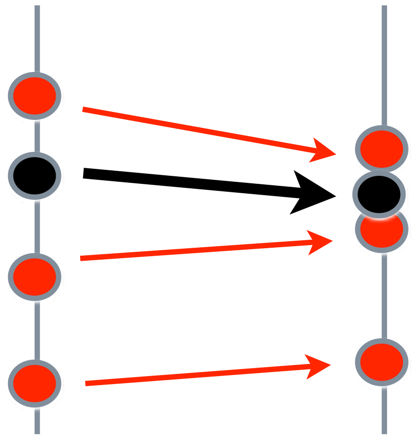

extension property

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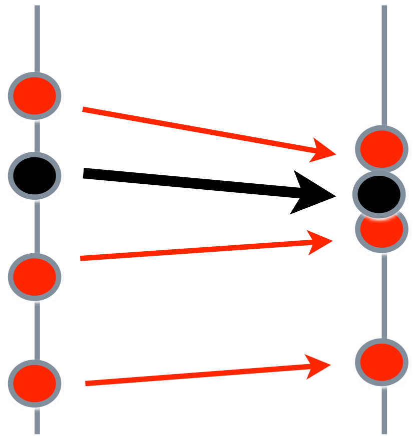

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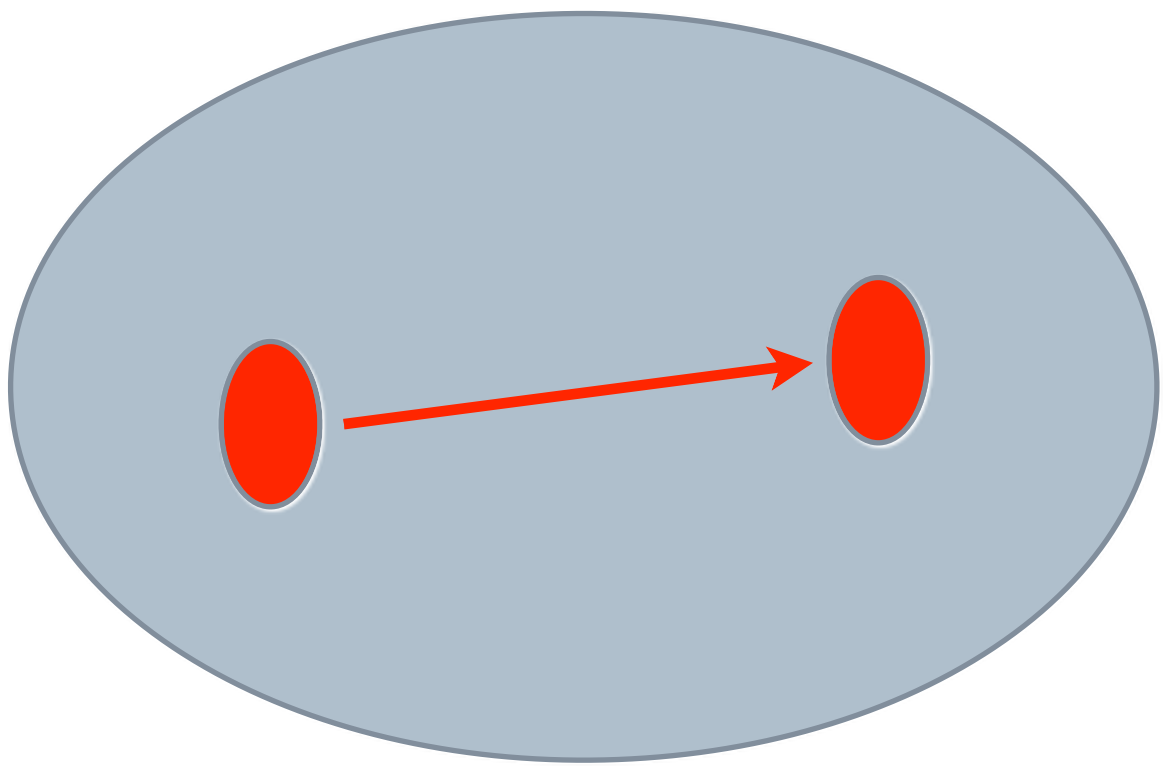
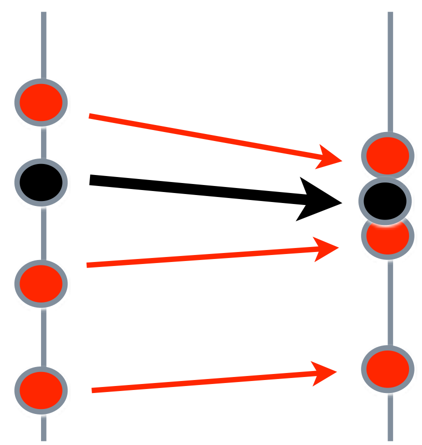
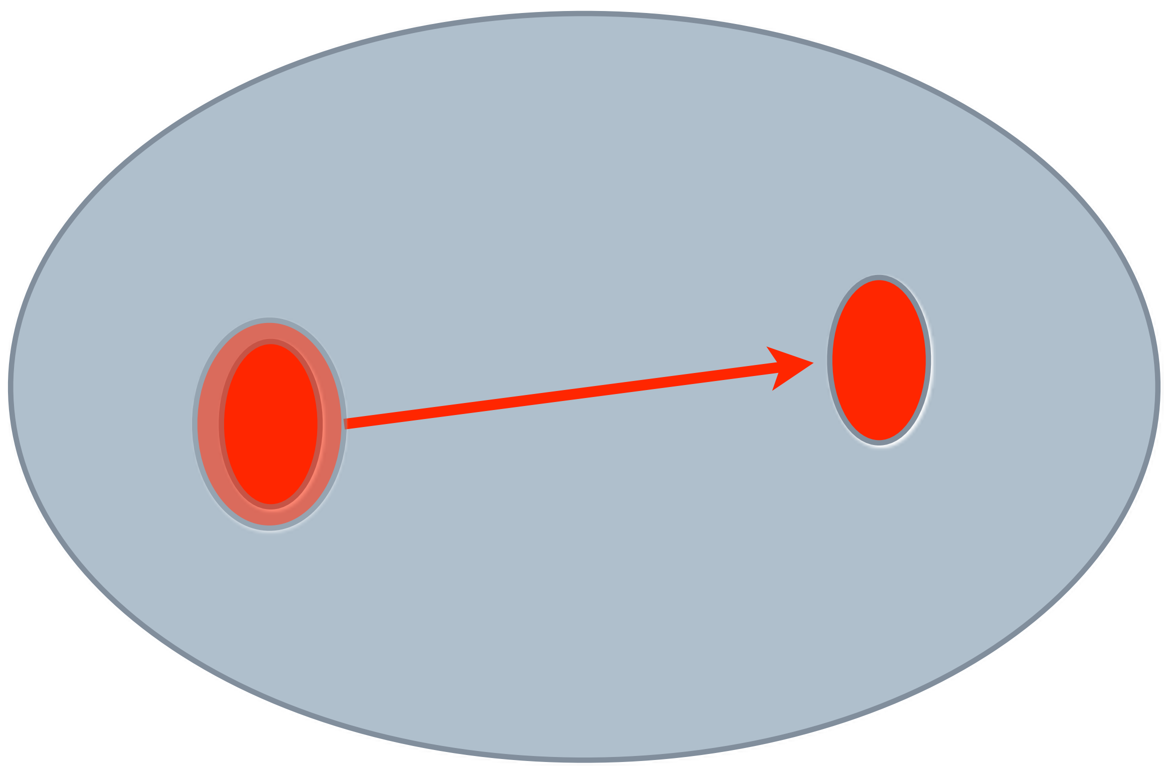


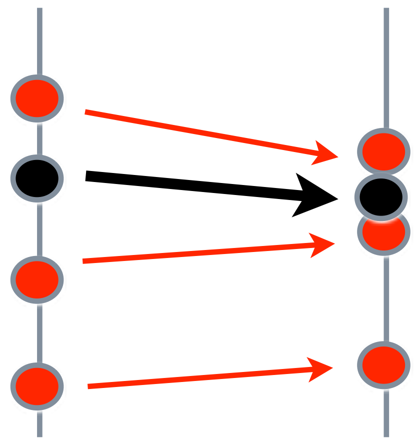
Homogeneous atoms (relational case)



extension property



Homogeneous atoms (relational case)



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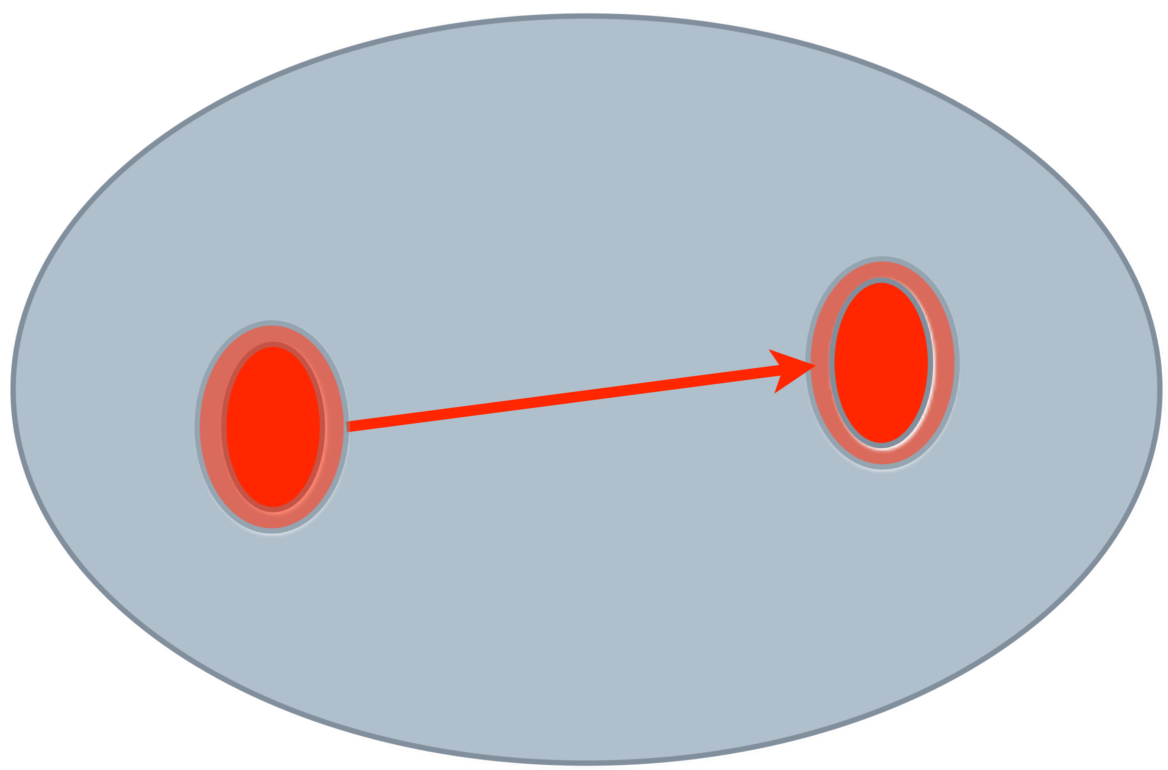

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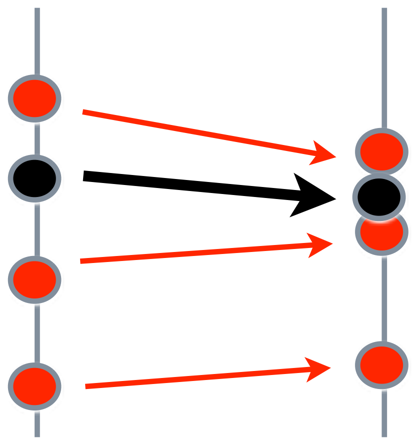

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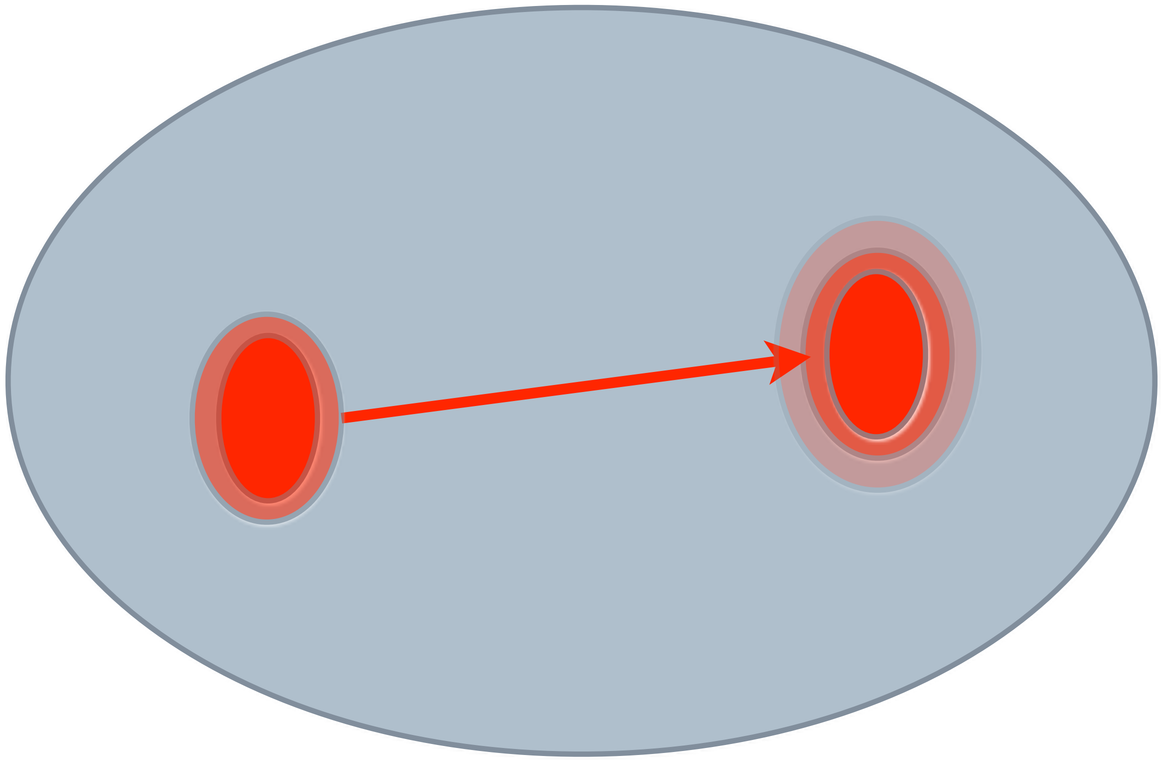

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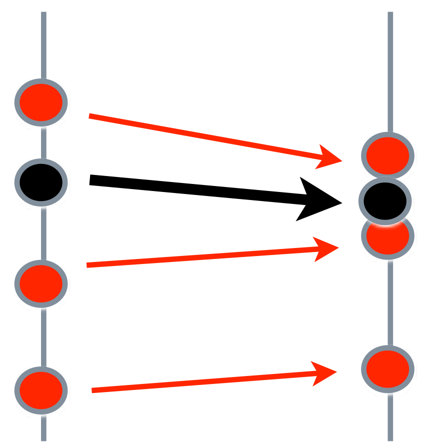

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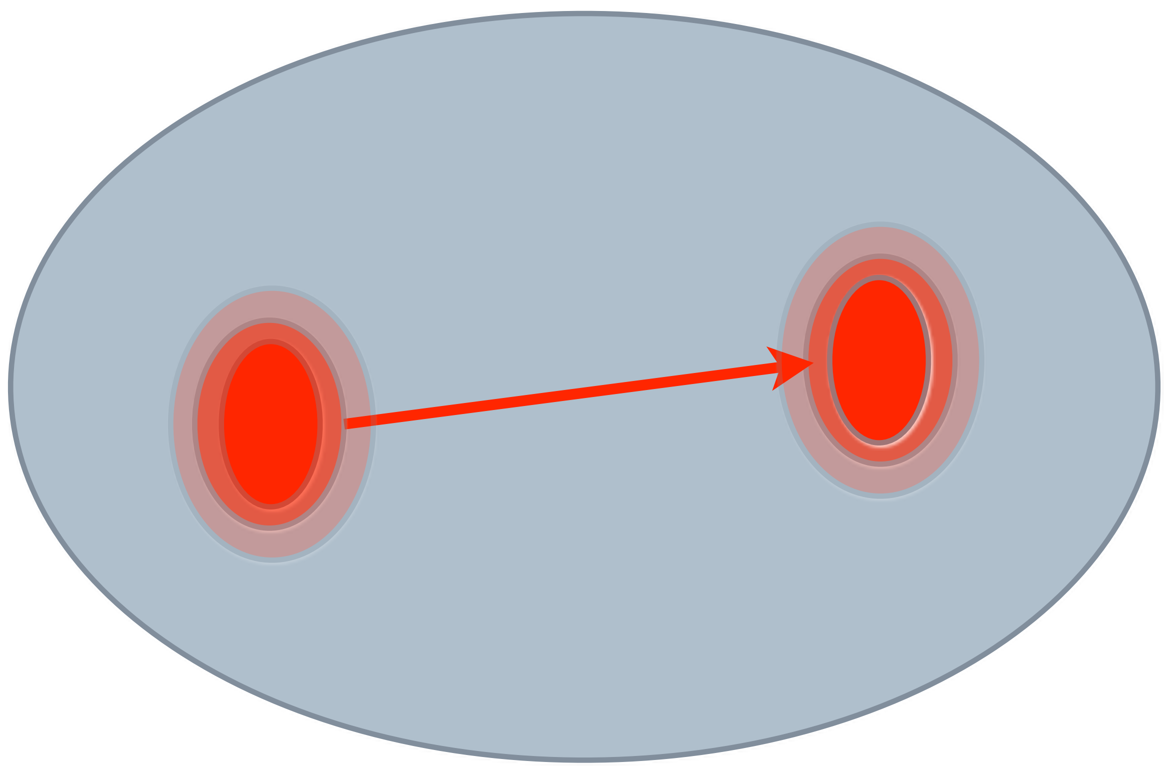

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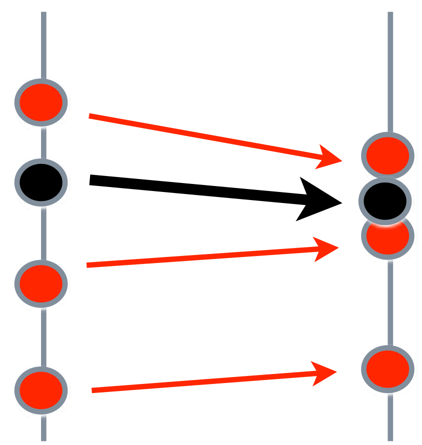

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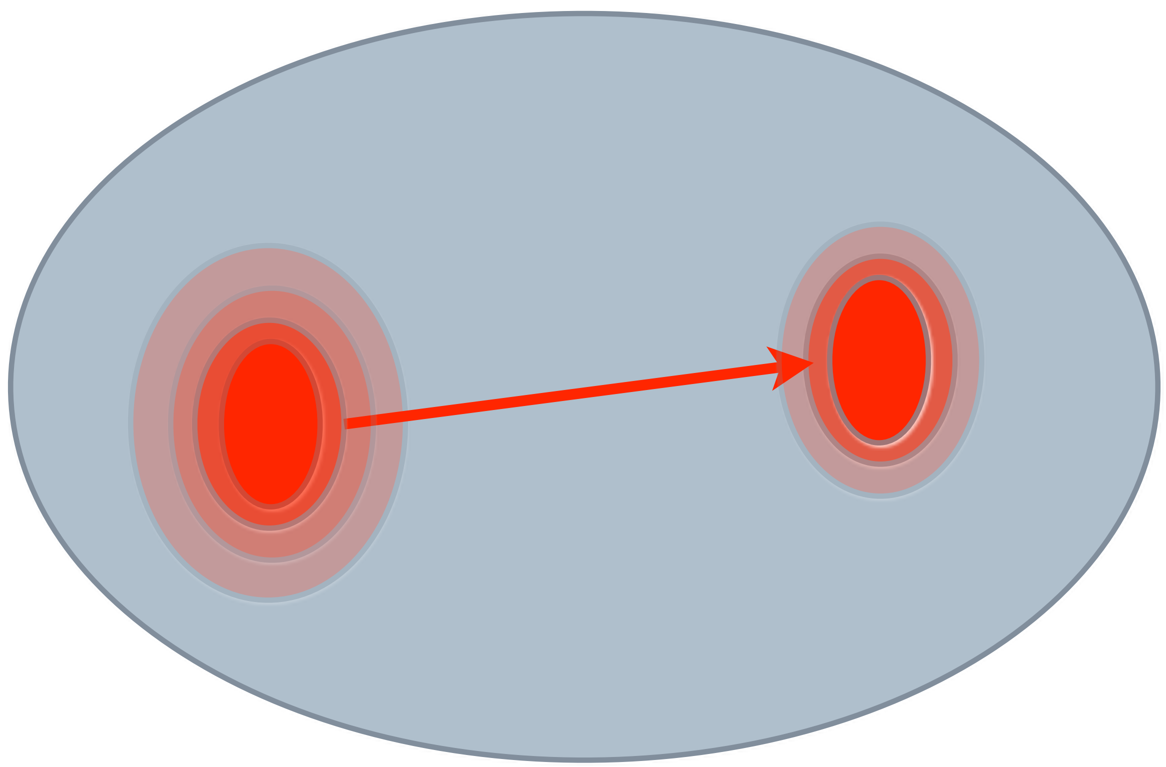
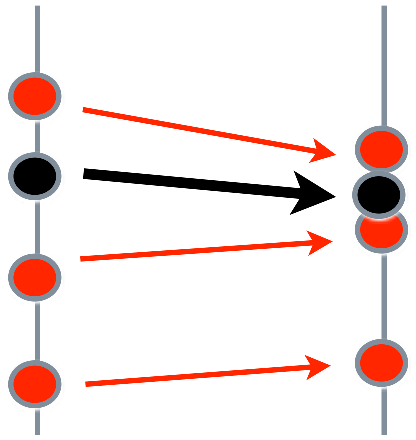
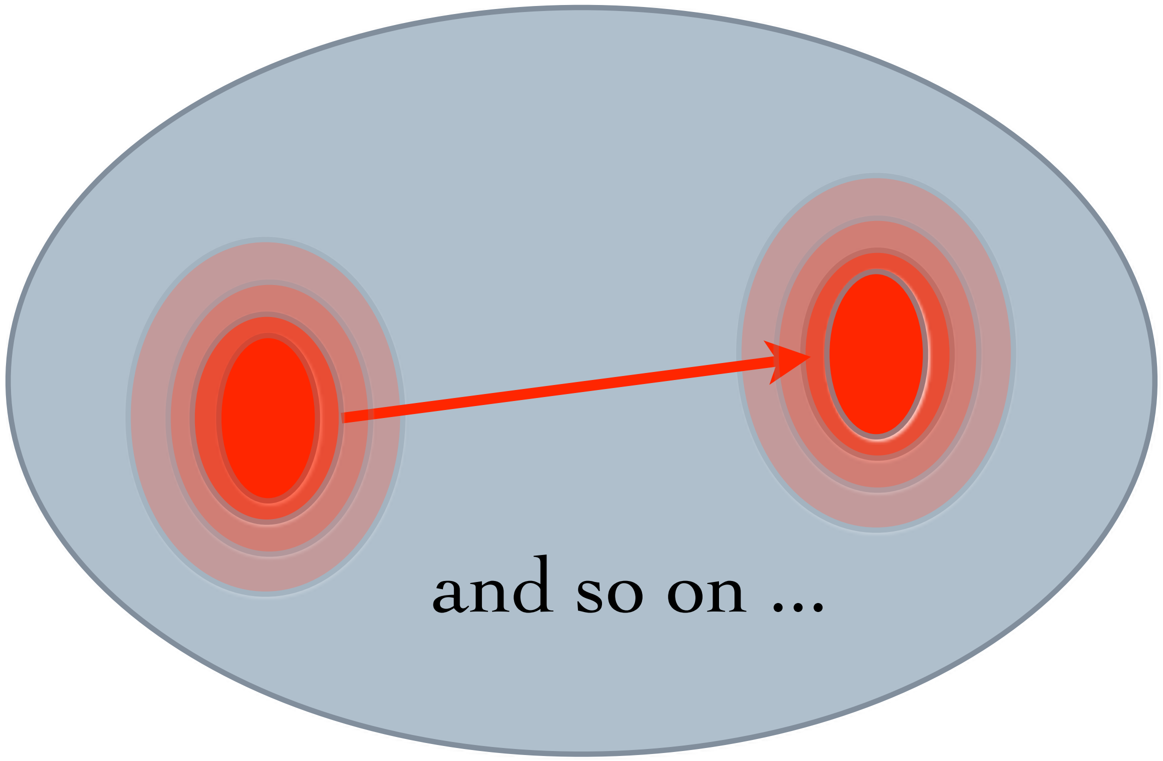


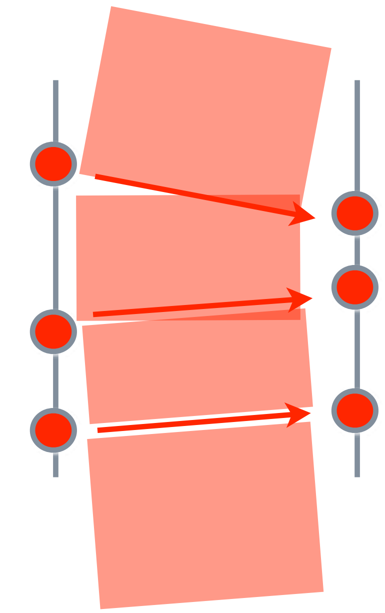
Homogeneous atoms (relational case)



extension property



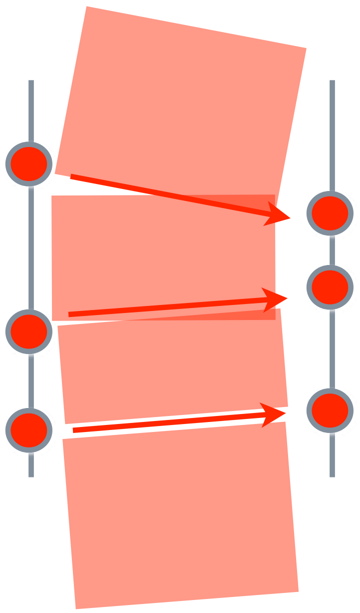
Quantifier elimination



Observation: When atoms are homogeneous

two tuples in atoms$^{(n)}$ are in the same orbit	iff	the induce generate isomorphic substructures

Quantifier elimination



Observation: When atoms are homogeneous

two tuples in atoms$^{(n)}$ are in the same orbit **iff** the induce generate isomorphic substructures

Corollary:

invariant subsets of $\mathbb{A}^n$ = **quantifier-free** definable subsets of $\mathbb{A}^n$