Lower bounds for reachability in VASS in fixed dimension

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Plan

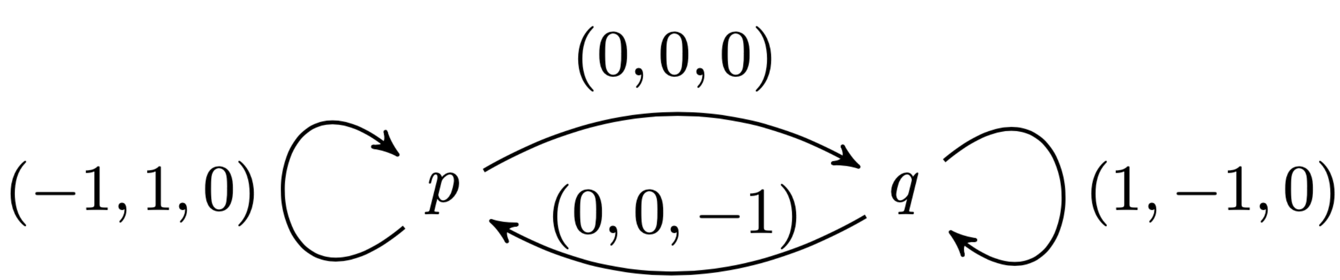
1. Vector addition systems with states (**VASS**) and the reachability problem

2. Lower bounds in **small** fixed dimensions

3. Lower bounds in **large** fixed dimensions

Many faces of vector addition systems with states

- vector addition systems with states **VASS** **[Hopcroft, Pansiot '79]**:

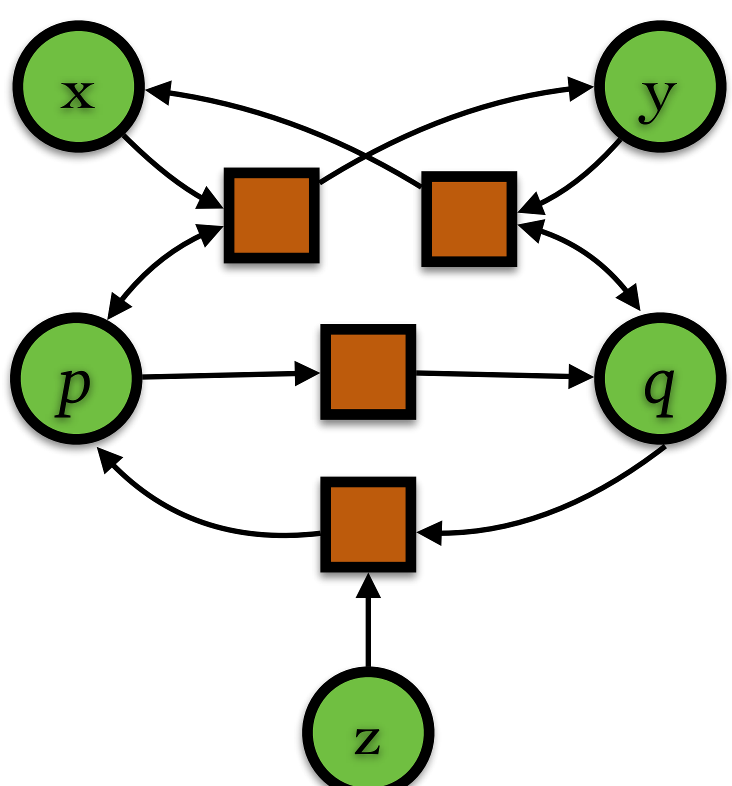


=

- counter programs **without 0-tests**:

```
loop
    loop
        x −= 1    y += 1
    loop
        x += 1    y −= 1
    z −= 1
```

- Petri nets **[Petri 1962]**:



- VAS **[Karp, Miller '69]**
- multiset rewriting
- …

Counter programs without zero tests

counters are nonnegative integer variables initially all equal zero

a sequence of commands of the form:

$\mathsf{x} \mathrel{+}= n$	(increment counter x by n)	
$\mathsf{x} \mathrel{-}= n$	(decrement counter x by n)	abort if $x < n$
goto L **or** L'	(jump to either line L or line L')	

except for the very last command which is of the form:

halt if $\mathsf{x}_1, \ldots, \mathsf{x}_l = 0$	(terminate provided all the listed counters are zero)	otherwise abort

Example:

1: $\mathsf{x}' \mathrel{+}= 100$
2: **goto** 5 **or** 3
3: $\mathsf{x} \mathrel{+}= 1 \quad \mathsf{x}' \mathrel{-}= 1 \quad \mathsf{y} \mathrel{+}= 2$
4: **goto** 2
5: **halt if** $\mathsf{x}' = 0$.

no zero tests

initially: $x' = x = y = 0$

finally: $x' = 0 \quad x = 100 \quad y = 200$

Loop programs

```
1: x' += 100
2: goto 5 or 3
3: x += 1    x' -= 1    y += 2
4: goto 2
5: halt if x' = 0.
```

```
1: x' += 100
2: loop
3:     x += 1    x' -= 1    y += 2
4: halt if x' = 0.
```

Minsky machines

the conditional jump of Minsky machines

$$\textbf{if } \mathsf{x} = 0 \textbf{ then goto } L \textbf{ else } \mathsf{x} \mathrel{-}= 1$$

is simulated by counter program **with zero tests**:

```
1: goto 2 or 4
2: zero? x
3: goto L
4: x -= 1
```

Reachability (and coverability)

Reachability problem: given a counter program **without zero tests**

```
1: x' += 100
2: goto 5 or 3
3: x += 1    x' −= 1    y += 2
4: goto 2
5: halt if x' = 0.
```

configuration reachability

can it terminate (execute its halt command)?

Coverability problem: given a counter program **without zero tests** **with trivial halt command**

```
1: x' += 100
2: goto 5 or 3
3: x += 1    x' −= 1    y += 2
4: goto 2
5: halt.
```

control-location reachability

can it terminate (reach its halt command)?

Complexity of reachability (and coverability)

Time/space needed is at least $F_3(n) = \left.2^{2^{2^{\cdots^{2}}}}\right\}O(n)$

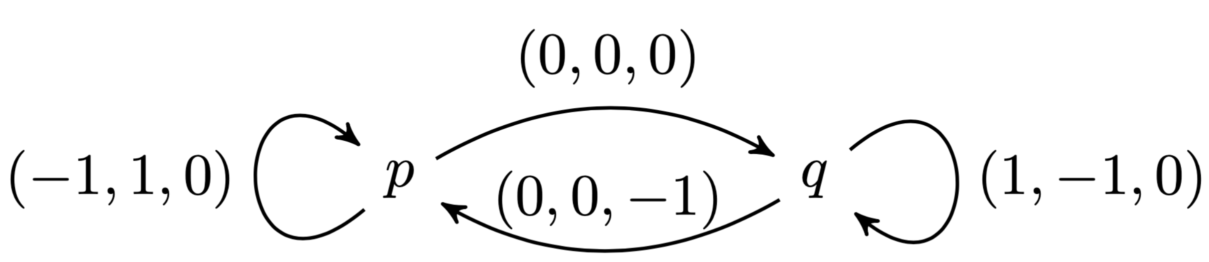
	coverability	reachability
lower bound	EXPSPACE [Lipton '76]	**Tower** [Czerwiński, L., Laz… …oux, Mazowiecki '19] ~~EXPSPACE~~ [Lipt…
upper bound	EXPSPACE [Rackoff '78]	A… …eroux, Schmitz '15, '19] ~~decidable~~ [Sacerdote, Tenney '77] [Mayr '81] [Kosaraju '82] [Lambert '92] [Leroux '09]

see Jerome Leroux's invited talk on Thursday

Time/space needed is at most $F_\omega(n)$

2. Lower bounds for reachability in **small** fixed dimensions

dimension = number of counters:




```
loop
    loop
        x −= 1    y += 1
    loop
        x += 1    y −= 1
    z −= 1
```

Reachability in dimension 1 and 2

encoding of integers

	unary	binary
dimension 1	NL* [Valiant, Peterson '75]	NP* [Haase, Kreutzer, Ouaknine, Worrell '09]
dimension 2	NL* [Englert, Lazic, Totzke '16]	PSPACE* [Blondin, Finkel, Goeller, Haase, McKenzie '15]

*complete

shortest run has **polynomial** length

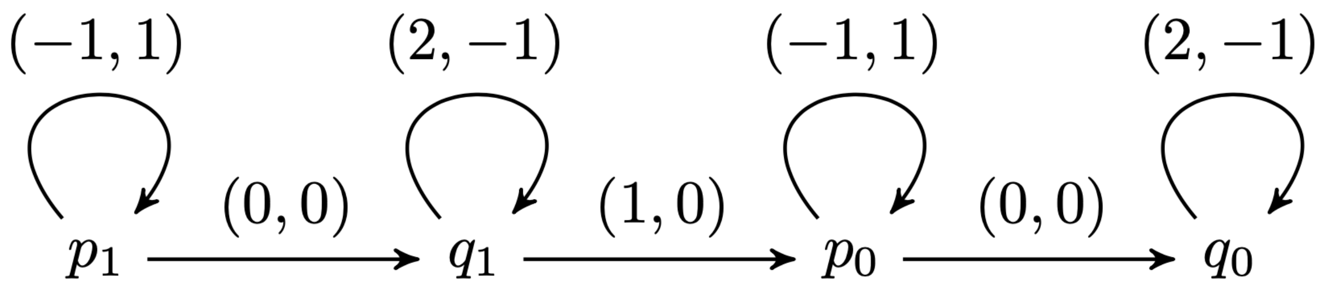
shortest run has **exponential** length

effectively **flattable** in dimension 2

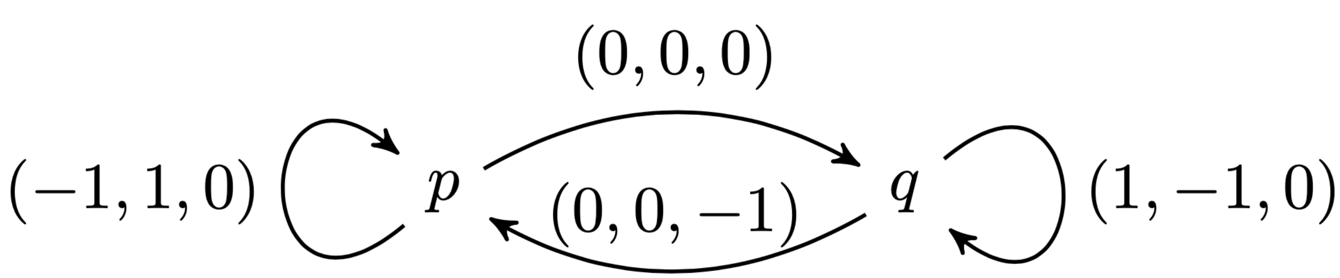
Upper bounds similar to dimension 2 for **every** fixed dimension?

…at least for flat counter programs?

Flat = no nested loops




```
loop
    x -= 1    y += 1
loop
    x += 2    y -= 1
x += 1
loop
    x -= 1    y += 1
loop
    x += 2    y -= 1
```




```
loop
    loop
        x -= 1    y += 1
    loop
        x += 1    y -= 1
    z -= 1
```

Shortest runs in small dimensions

[Czerwiński, L., Lazic, Leroux, Mazowiecki '20]

	unary flat	unary	binary
dim 1	poly*	poly*	exp*
dim 2	poly*	poly*	exp*
dim 3	**exp***	exp*	**?**
dim 4			**2-exp***
dim 5			
…			

*upper bound
*lower bound

Key ingredient: computing **exactly** large numbers

Exponential shortest run in unary flat dim 3

```
x += 1    y += 1
loop
    x += 1    y += 1
for i := n down to 1 do
    loop
        x −= 1    z += 1
    loop
        x += i+1    z −= i
loop
    x −= n+1    y −= 1
halt if y = 0.
```

objective:

halts iff y divisible by L

halts iff x = $(n+1) \cdot y$

iff all multiplicatio

iff all inner loops it

iff $(n+1) \cdot y$ divisi

$$x <= \frac{n+1}{n} \cdot \frac{n}{n-1} \cdot \ldots \cdot \frac{3}{2} \cdot \frac{2}{1} \cdot y$$

```
loop
    x −= 1    z += 1
loop
    x += 5    z −= 4

loop
    x −= 1    z += 1
loop
    ...
    x −= 1    z += 1
loop
    x += 3    z −= 2

loop
    x −= 1    z += 1
loop
    x += 2    z −= 1
```

program size $O(n^2)$, shortest run $2^{O(n^c)}$

Complexity lower bounds for reachability

[Czerwiński, L., Lazic, Leroux, Mazowiecki '19, '20]
[Dubiak '20]

What about coverability?

	unary flat	unary	binary
dim 1	NL*	NL*	NP*
dim 2	NL* NL	NL* NL	PSPACE* PSPACE
dim 3	? NL	NL	PSPACE
dim 4	NL	NL ?	PSPACE
dim 5	NP* NL	NL	PSPACE
…	NP* NL	NL	PSPACE
dim 13	NP* NL	PSPACE* NL	EXPSPACE* PSPACE
dim 14	NP* NL	EXPSPACE* NL	2-EXPSPACE* PSPACE
dim 15	NP* NL	2-EXPSPACE* NL	3-EXPSPACE* PSPACE
…	… NL	… NL	… PSPACE

also for binary flat

*complete
*hard

3. Lower bounds for reachability in **large** fixed dimensions

Parametric lower bound for reachability

[Czerwiński, L., Lazic, Leroux, Mazowiecki '19]
[Czerwiński, L., Orlikowski '??]

	unary flat	unary	binary
dim 13		**PSPACE***	
dim 14		**EXPSPACE***	
dim 15		**2-EXPSPACE***	
...		...	

Minsky machine $\mathcal{M}$ of size n
with counters bounded by

$$\left.2^{2^{2^{\cdots^{n}}}}\right\}h+1$$

$\mathcal{O}$(n+h) ⟶ Counter program of size $\mathcal{O}$(n+h) with $\mathcal{O}$(h) = h+13 counters

$\mathcal{O}$(n+log h), improved reduction ⟶ Counter program of size $\mathcal{O}$(n+log h) with $\mathcal{O}$(log h) counters

Parametric lower bound for reachability

[Czerwiński, L., Lazic, Leroux, Mazowiecki '19]
[Czerwiński, L., Orlikowski '??]

	unary flat	unary	binary
dim 13		**PSPACE***	
dim 14		**EXPSPACE***	
dim 15		**2-EXPSPACE***	
...		...	

Reachability problem for programs of size n with d counters needs space at least

$$2^{2^{2^{\cdots^{n}}}} \Big\} O(d) \qquad d-13$$

improved lower bound:

$$2^{2^{2^{\cdots^{n}}}} \Big\} 2^{O(d)} \qquad 2^{(d-13)/3}$$

Exponential amplifier

```
i += 1    x += 1    y += 1    b += 1    c += 1    d += 1
loop
    x += 1    y += 1    c += 1    d += 1
loop
    loop
        c −= i    c′ += 1
        loop at most b times
            x −= i    d −= i    x′ += i + 1
    loop
        b −= 1    b′ += i + 1
    loop
        b′ −= 1    b += 1
    loop
        c′ −= 1    c += 1
        loop at most b times
            x′ −= 1    x += 1    d += 1
    i += 1
zero? î
loop
    x −= i    y −= 1
halt if y = 0
```

```
loop
    b −= 1    b′ += 1
loop
    b′ −= 1    b += 1
    <body>
```

```
x′ += 1
loop
    i −= 1    i′ += 1    x′ += 1
zero? i
loop
    i′ −= 1    i += 1
zero? i′
```

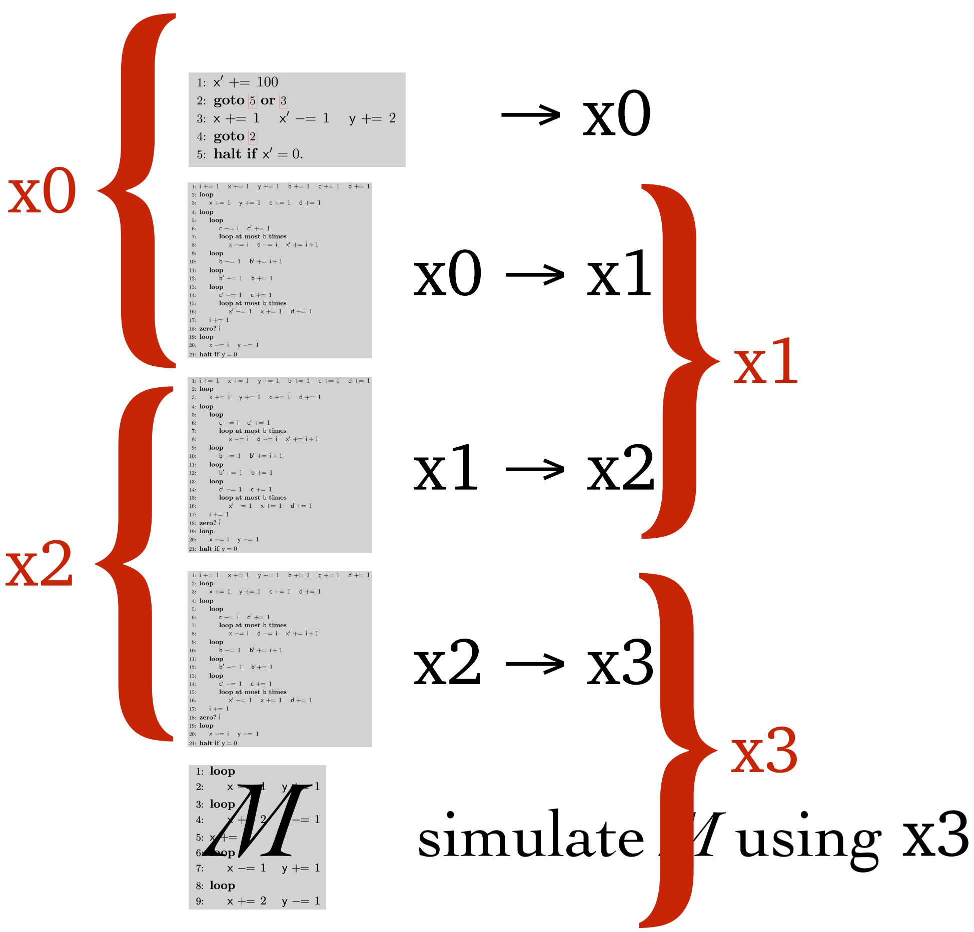
```
loop
    i −= 1    i′ += 1    x −= 1
zero? i
loop
    i′ −= 1    i += 1
zero? i′
```

x → x′: starting with x>0 and x′ = 0, computes x′ exponentially larger than x (if x = 0 at the end)

if so, also all other counters are forcedly 0 at the end

Composing exponential amplifiers

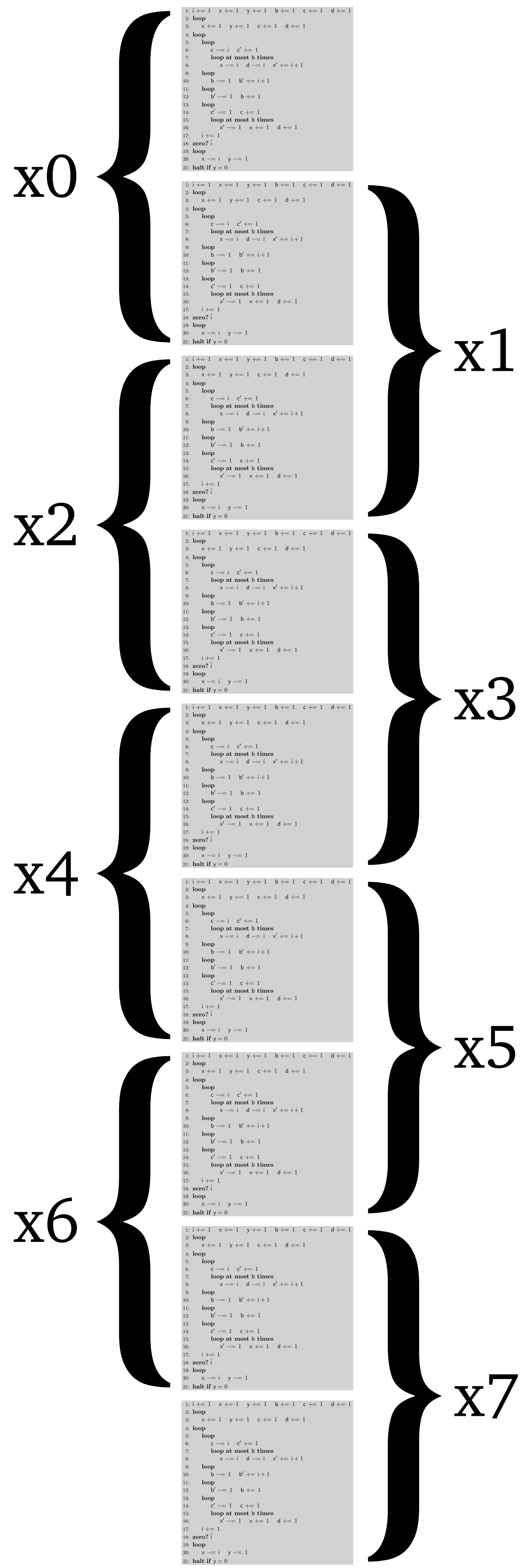
	unary flat	unary	binary
dim 13		PSPACE*	
dim 14		EXPSPACE*	
dim 15		2-EXPSPACE*	
…		…	



halt if x0, x1, x2, x3 = 0

relay-race

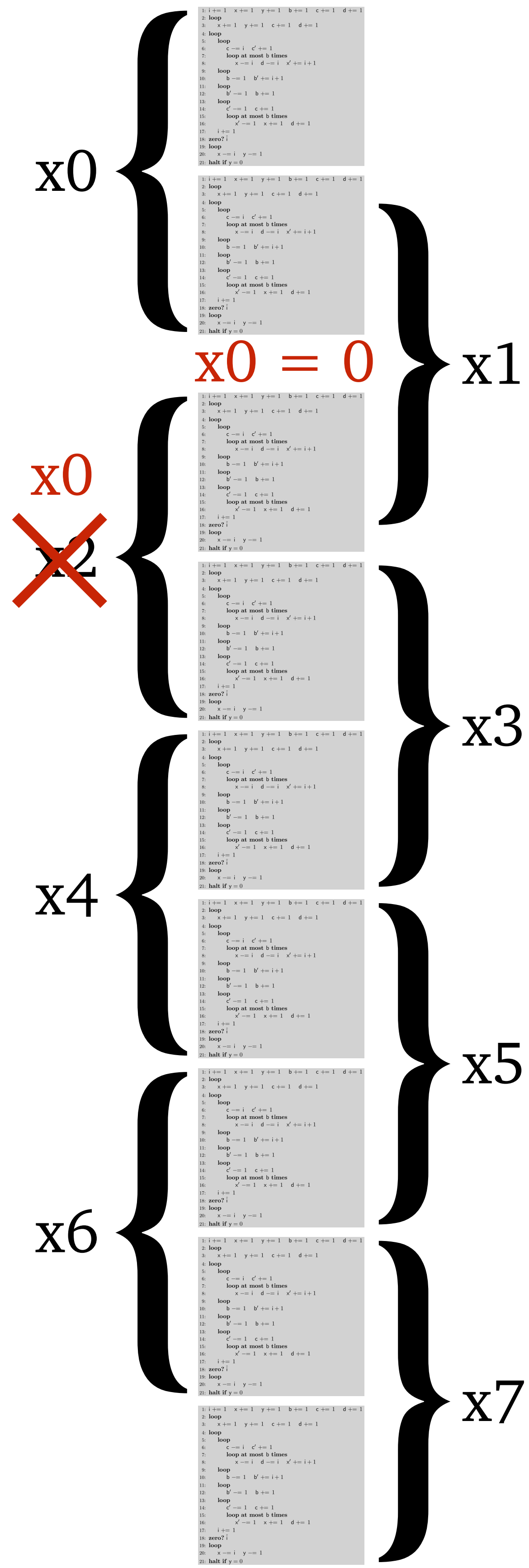
Relay-race



objective: decrease the number of counters to logarithmic

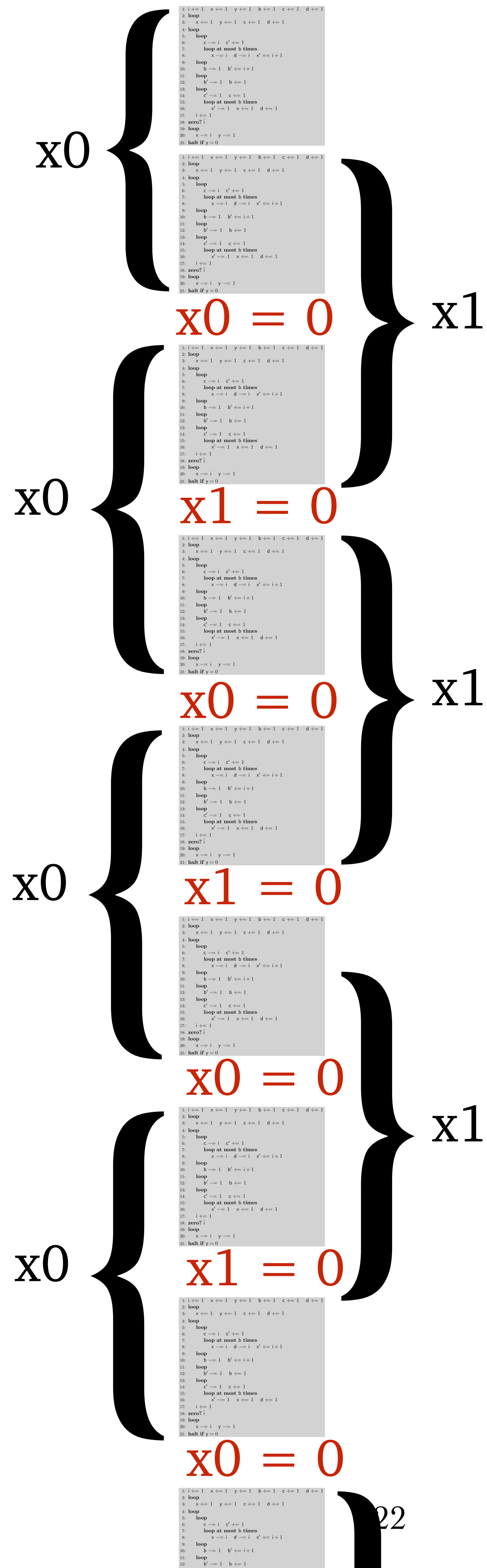
halt if x0, x1, x2, x3, x4, x5, x5, x7 = 0

Counter recycling?

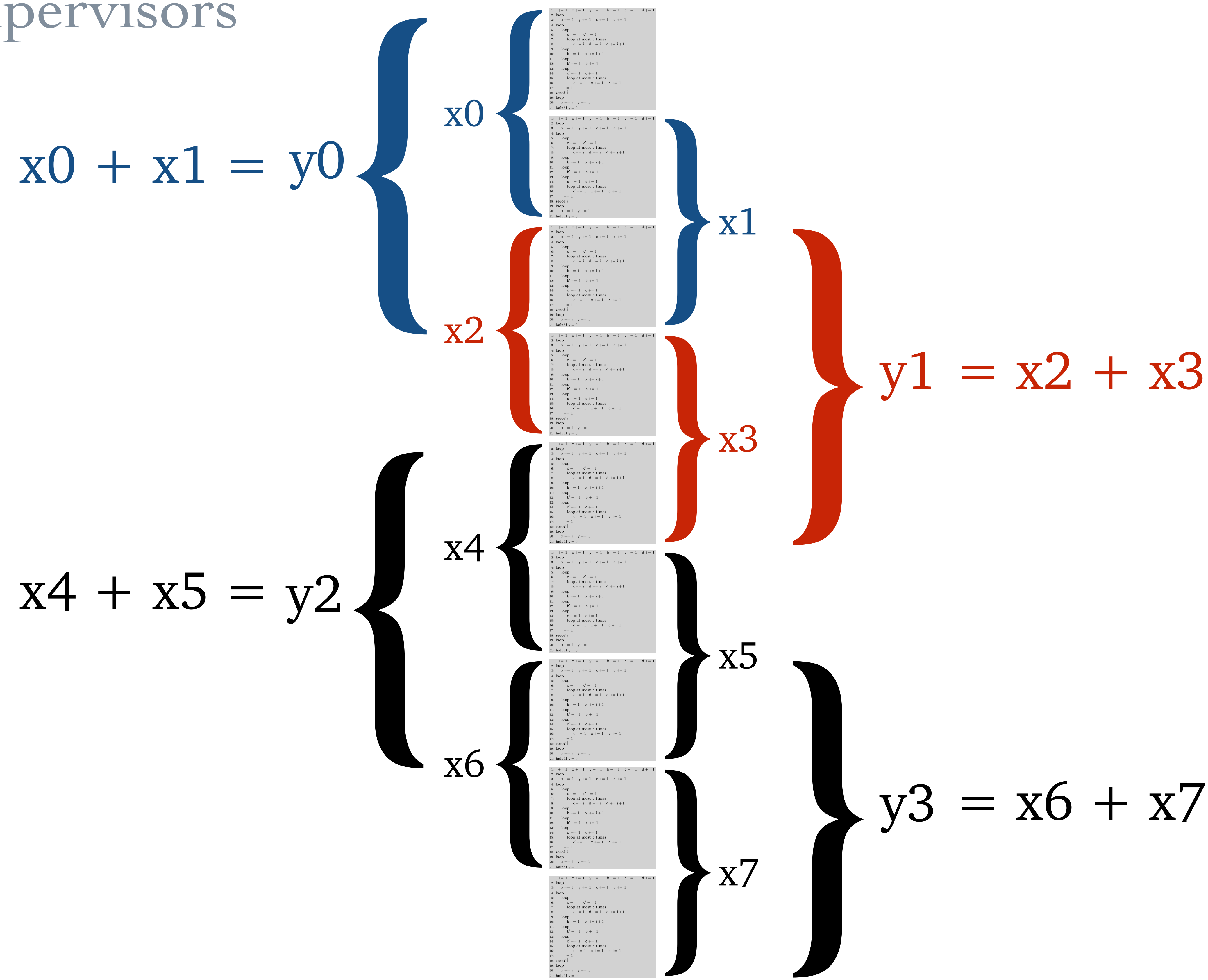


halt if x0, x1, ~~x2~~, x3, x4, x5, x5, x7 = 0

Counter recycling?

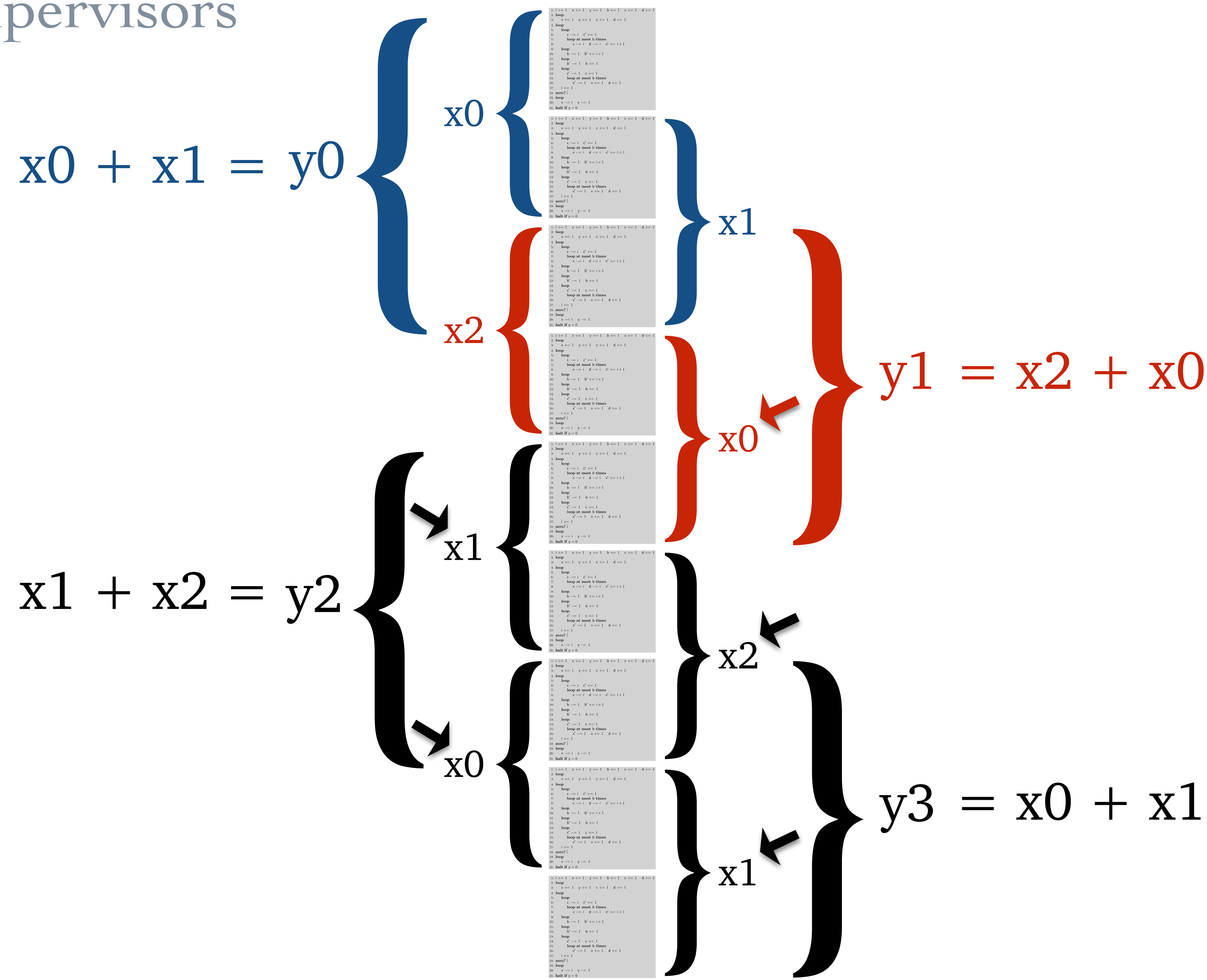


Supervisors



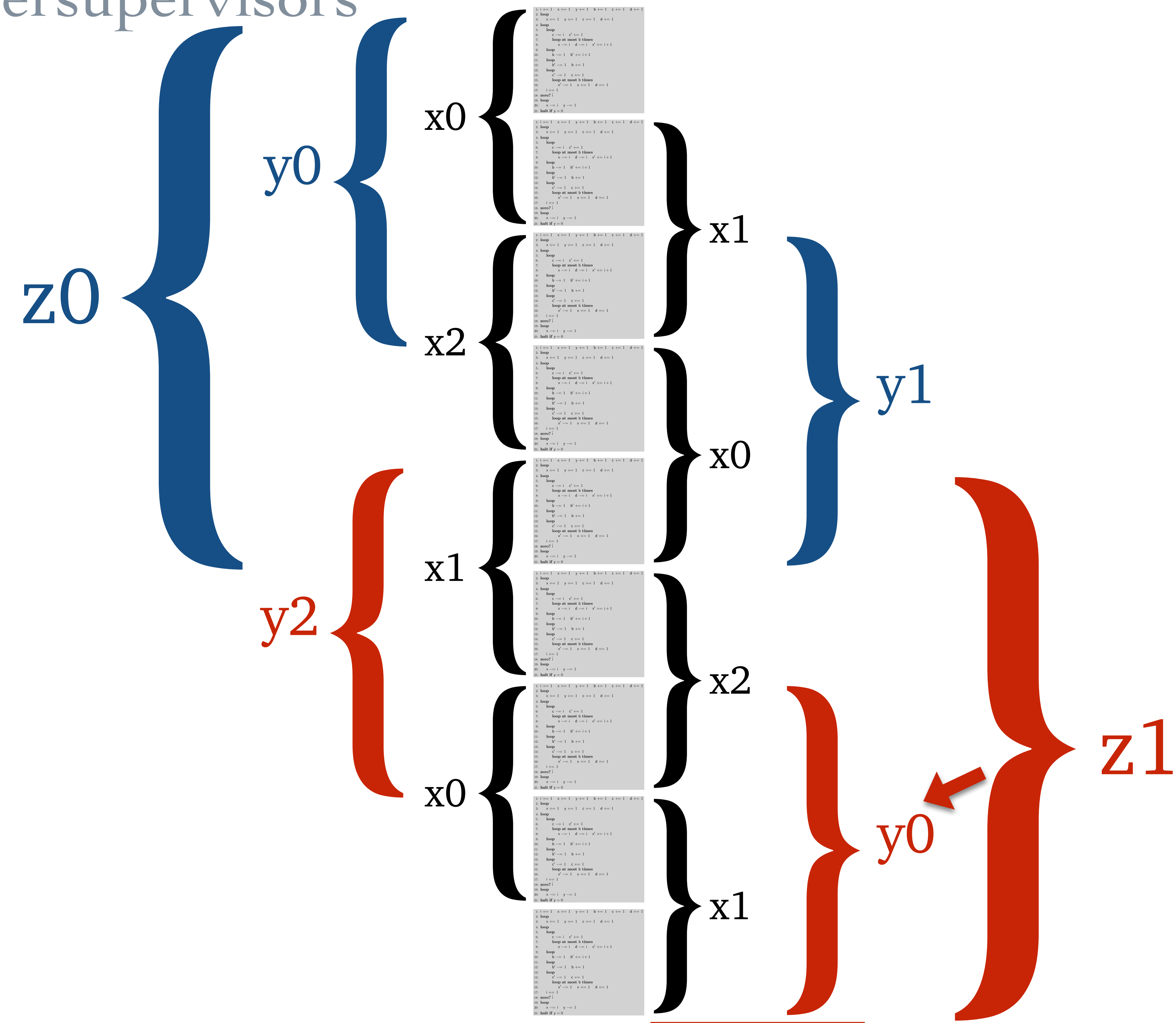
halt if x0, x1, x2, x3, x4, x5, x5, x7,
y0, y1, y2, y3, … = 0

Supervisors



halt if x0, x1, x2, y0, y1, y2, y3, … = 0

Supersupervisors



halt if x0, x1, x2,
y0, y1, y2,
z0, z1, … = 0

and so on…

Summary

	unary flat	unary	binary
dim 1	NL*	NL*	NP*
dim 2	NL*	NL*	PSPACE*
dim 3	?	?	?
dim 4	?	?	?
dim 5	NP*	?	?
…	NP*	?	?
dim 13	NP*	PSPACE*	EXPSPACE*
dim 14	NP*	EXPSPACE*	2-EXPSPACE*
dim 15	NP*	2-EXPSPACE*	3-EXPSPAC
…	…	…	…

*complete
*hard

	unary flat	unary	binary
dim 1	poly*	poly*	exp*
dim 2	poly*	poly*	exp*
dim 3	exp*	exp*	?
dim 4			2-exp*
dim 5			
…			

*upper bound
*lower bound

I recruit for a fully-funded PhD position in the NCN grant

Reachability problem for d-dimensional VASS of size n requires space at least

$$2^{2^{2^{\cdot^{\cdot^{\cdot^{n}}}}}}\Bigg\} 2^{O(d)}$$

thank you!