On Boundedness Problems for Pushdown Vector Addition Systems

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Vector Addition Systems – Recap

Definition

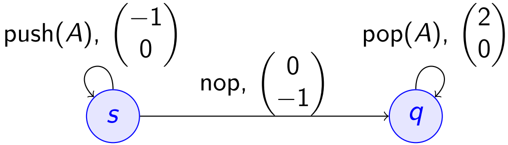
A VAS is a finite set of vectors $\mathbf{a} \in \mathbb{Z}^d$. For $\mathbf{v}, \mathbf{v}' : \mathbb{N}^d$ it has a step

$$\mathbf{v} \xrightarrow{\mathbf{a}} \mathbf{v}' \quad \text{if} \quad \mathbf{v}' = \mathbf{v} + \mathbf{a}.$$

- Equivalent to Petri Nets
  (concurrency, weak counters, event systems)
- Reachability: decidable
  Mayr'81,*Kosaraju'82*, . . . *Leroux and Schmitz'15*
- Coverability, Boundedness: ExpSpace-complete
  Lipton'76, *Rackoff'78*
- Most Games/Equivalences undecidable (e.g. Bisimulation)
  Jančar'95

Pushdown Vector Addition Systems

. . . are products of VAS with pushdown automata.



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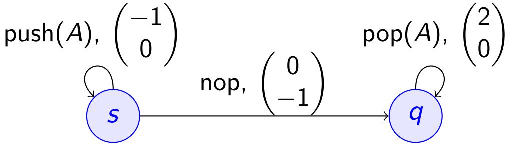

$s, \bot, \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

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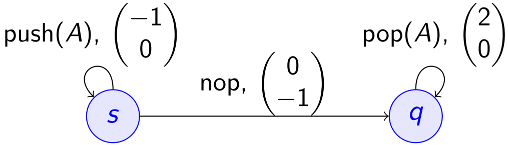

$$s, \perp, \begin{pmatrix}2\\1\end{pmatrix} \longrightarrow\longrightarrow s, AA\perp, \begin{pmatrix}0\\1\end{pmatrix}$$

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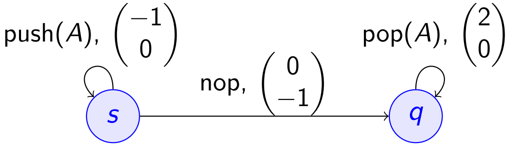

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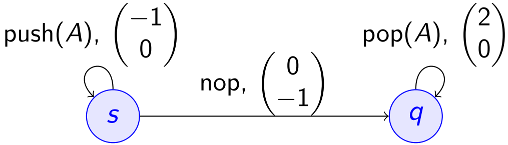

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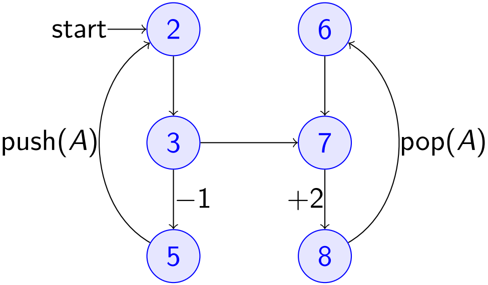
Pushdown Vector Addition Systems

. . . are products of VAS with pushdown automata. They can for example model recursive prorams with variables over $\mathbb{N}$.

```
x ← n
procedure DOUBLEX
    if (⋆ ∧ x > 0) then
        x ← (x − 1)
        DOUBLEX
    end if
    x ← (x + 2)
end procedure
```



Pushdown Vector Addition Systems

- Reachability $=$ Coverability $(=$ State-Reachability$)$
  TOWER-hard *Lazic'13*

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- Boundedness: decidable with Hyper-Ackermannian bounds
  Leroux, Praveen, and Sutre'14

Theorem [LSP'14]

If a PVAS configuration $(p, \perp, n)$ is bounded then the cardinality of the reachability set is at most $F_{\omega^d \cdot |Q|}(d \cdot n)$.

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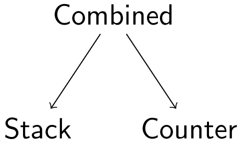

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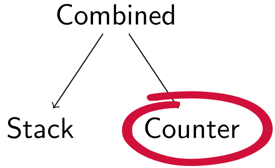

The following is in EXPTIME.

1-PVAS Counter-Boundedness

Given: 1-dim. PVAS, initial configuration (p, w, a).

Question: is $\{b \mid (p,w,a) \xrightarrow{*} (p', w', b)\}$ infinite?

Another Perspective

Definition (Context-free Controlled VAS)

a VAS $\mathbf{A} \subseteq \mathbb{Z}^d$ together with a context-free language $\mathcal{L} \subseteq A^*$.
There is a step $\mathbf{s} \longrightarrow \mathbf{t}$ between $\mathbf{s}, \mathbf{t} \in \mathbb{N}^d$ if

$$\mathbf{a_1 a_2} \ldots \mathbf{a_k} \in \mathcal{L} \quad \text{and} \quad \mathbf{s} \xrightarrow{\mathbf{a_1}} \xrightarrow{\mathbf{a_2}} \cdots \xrightarrow{\mathbf{a_k}} \mathbf{t}.$$

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For Cf-Controlled VAS, Coverability (and Reachability) logspace reduces to Boundedness.

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Relevant for the PVAS boundedness problem is the *trace* language $\{w \in \mathbf{A}^* \mid (p_0, \perp) \xrightarrow{w}\}$ defined by the PDA.

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prefix-closed!

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Main Theorem

Boundedness of 1-dim VAS controlled by a prefix-closed language is in ExpTime.

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given as GfG

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$$X \overset{*}{\Longrightarrow} \mathbf{a_1 a_2 \ldots a_k} \quad \text{and} \quad \mathbf{s} \xrightarrow{\mathbf{a_1}} \xrightarrow{\mathbf{a_2}} \cdots \xrightarrow{\mathbf{a_k}} \mathbf{t}.$$

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Flow Trees

A derivation tree with *consistent* in/out labels in $\mathbb{Z} \cup \{-\infty\}$.

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$\boxed{X}$

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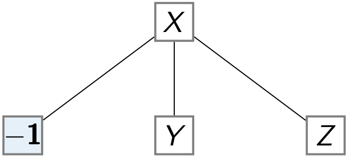

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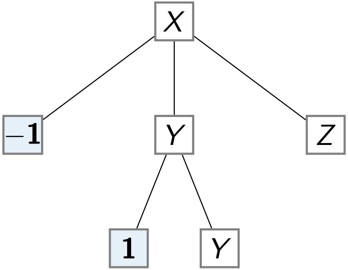

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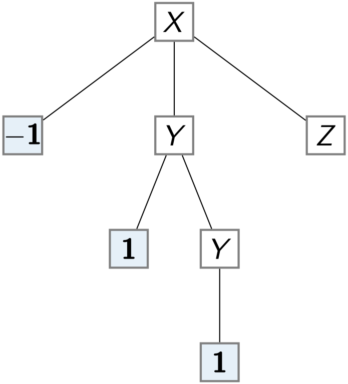

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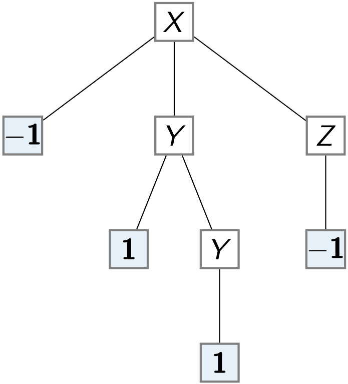

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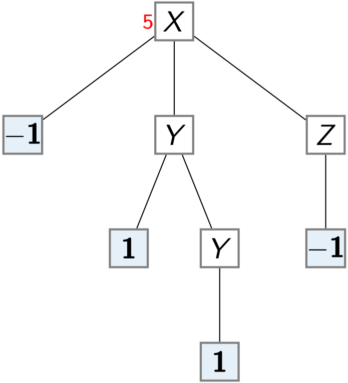

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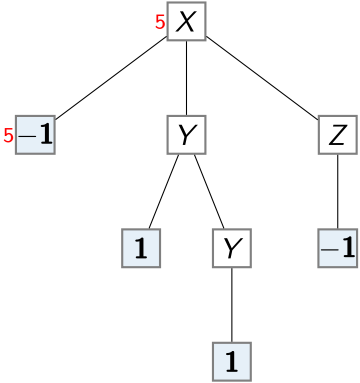

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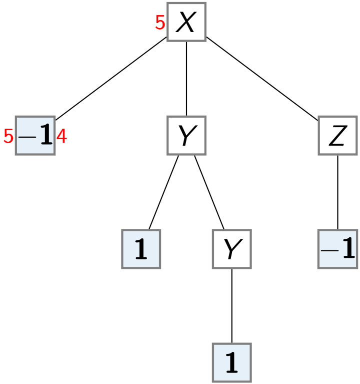

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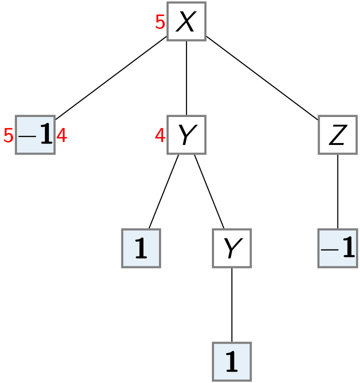

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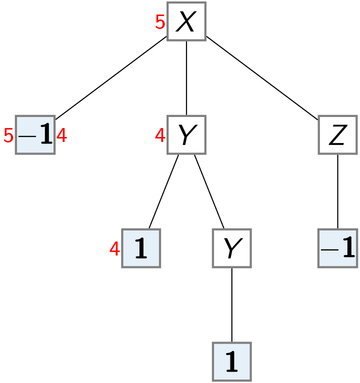

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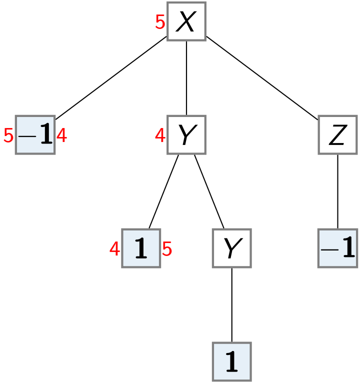

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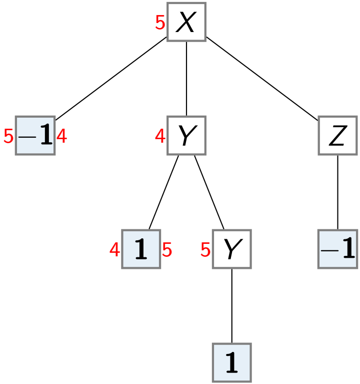

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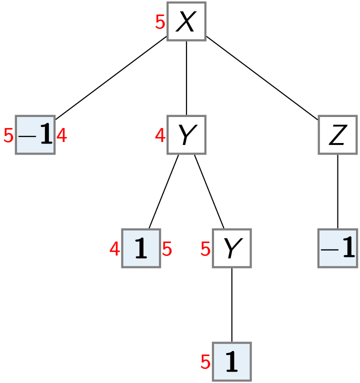

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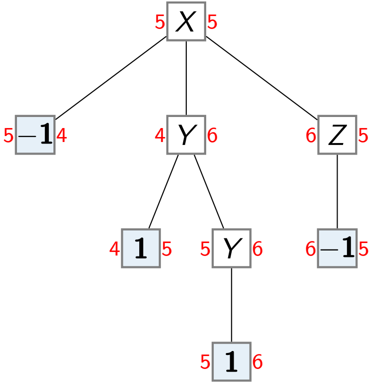

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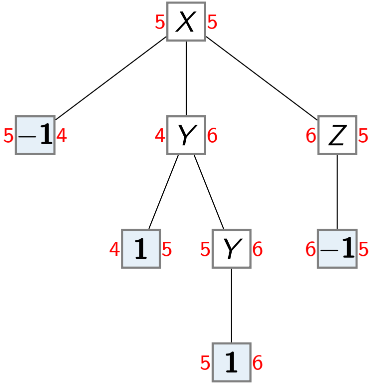

$_a\boxed{X}_b$ means $a \xrightarrow{X} b' \geq b$; $\quad {}_{-\infty}\boxed{X}_b$ means $\exists a \in \mathbb{N}.\ a \xrightarrow{X} b' \geq b$.

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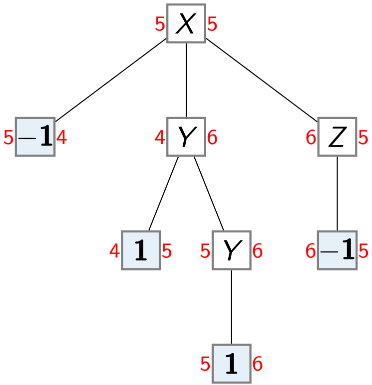

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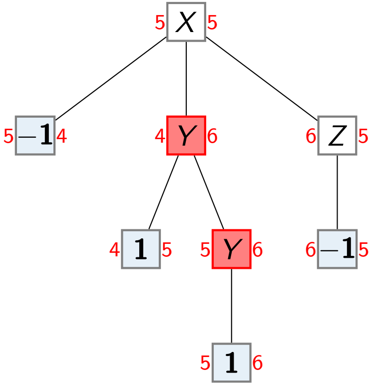

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Certificates

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A *certificate* is a flow tree with a node $_b\boxed{X}_{b'}$ and a descendant $_c\boxed{X}_{c'}$ such that

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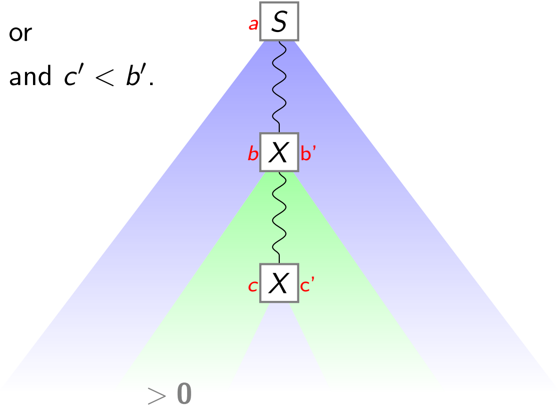

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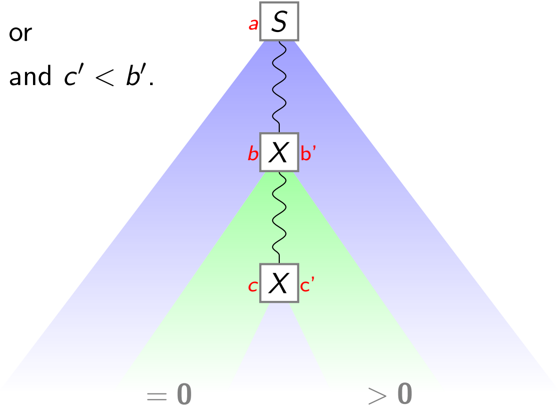

Certificates (cont.)

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$\{a' \mid a \xrightarrow{S} a'\}$ *is infinite iff there is a certificate with root* $(\leq a)$ $\boxed{S}$.

Unboundedness $\implies$ Certificate:

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Bounding $\sqsubset$-minimal Certificates

(maybe on blackboard if time)

Theorem

Let $G = (V, \mathbf{A}, R, S)$ be a CfG generating a prefix-closed language over $\mathbf{A} = \{-1, 0, 1\}$ and $n \in \mathbb{N}$ an initial value. Then $\{m \mid n \xrightarrow{S} m\}$ is infinite iff it admits a certificate with height and all input/output values bounded by $n + 4^{4(|V|+1)}$.

Conclusion

Discussed here

- Pushdown VAS; Boundedness of counter/stack/both
- Cf-controlled VAS; Flow Trees
- prefix-closed control $\sim$ counter-Boundedness
- Counter-Boundedness in 1-PVAS is in $\textsc{ExpTime}$

Open Problems

- Decidability of PVAS Reachability (even in dim 1)
- is Boundedness reducible to Reachability in Cf-C-VAS?
- Complexity of 1-PVAS counter-Boundedness (NP– $\textsc{ExpTime}$)
- Complexity of 1-PVAS Coverability (NP– $\textsc{ExpSpace}$)

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Additional Stuff

Weak Computation of Ackermann Functions A_m

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$$\begin{aligned} A_0(n) &= n+1 \\ A_1(n) &= n+2 \\ A_2(n) &= 2n+2 \\ A_3(n) &= 2^n - 1 \\ &\vdots \end{aligned}$$

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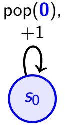

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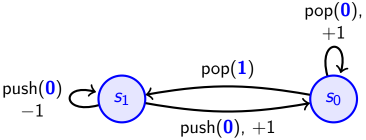

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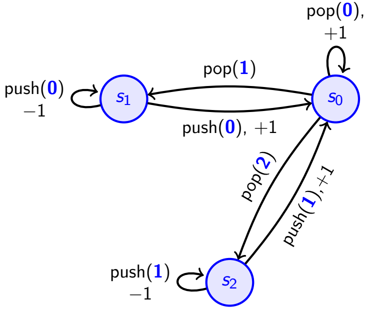

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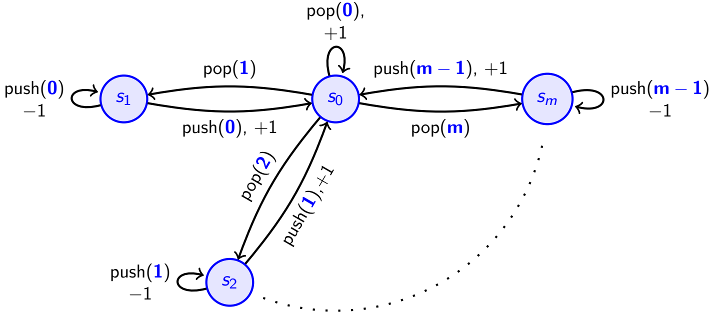

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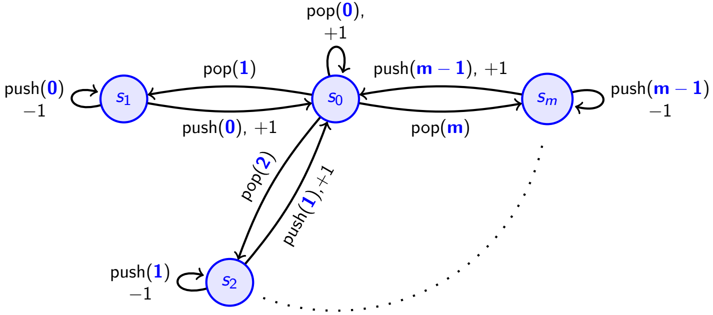

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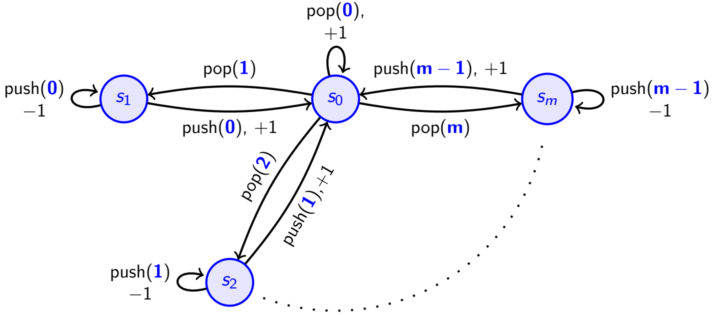

$$(s_0, \mathbf{m}\bot, n) \xrightarrow{*} (s_0, \bot, A_m(n))$$

If $(s_0, \mathbf{m}\bot, n) \xrightarrow{*} (s_0, \bot, n')$ then $n' \leq A_m(n)$